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Aggregation Across Each Nation: Aggregator Choice and Macroeconomic Dynamics

Noëmie Lisack[1], Simon Lloyd[2], Rana Sajedi[2]

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Abstract

We study the implications of trade aggregation in an infinite-horizon economy with multiple countries, asking whether there is a role for alternatives to the Armington aggregator in a wide range of workhorse open-economy macroeconomics models. We show analytically that the first-order dynamics of the model are entirely captured by a few sufficient statistics. Over and above these statistics, the precise choice of functional form for the trade aggregator is irrelevant. This result has the following implications. For given steady-state trade elasticities and expenditure shares, any aggregator that is homogeneous of degree one is equivalent to the Armington aggregator at first order. Similarly, aggregators that are homogeneous of arbitrary degree are equivalent to a simple generalisation of the Armington aggregator, for given steady-state trade elasticities and expenditure shares. In models with more than two countries, alternative aggregators can play a role by allowing for steady-state differences in bilateral trade elasticities across different country pairs, which the Armington aggregator rules out.

Keywords: International Trade, Open-economy Macroeconomics, Armington Aggregator, Elasticity of Trade.

JEL classification: F00, F10, F41.

[1] Banque de France, noemie.lisack@banque-france.fr .

[2] Bank of England, simon.lloyd@bankofengland.co.uk, rana.sajedi@bankofengland.co.uk

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NON-TECHNICAL SUMMARY

Goods trade is central to structural international macro models, including those used for policy analysis (e.g. the GIMF model used at the International Monetary Fund). One of the most primitive assumptions needed within these models is how agents allocate their consumption across domestic and foreign goods, i.e. how these goods are 'aggregated' together. The precise nature of this aggregation determines the make-up of cross-border trade, and can affect how demand responds to changes in international relative prices.

The 'go-to' choice of aggregator within workhorse international policy models is the constant elasticity of substitution (CES) aggregator, which is governed by two parameters. First, the share of expenditure on each good, which captures 'home bias'—the idea that countries tend to spend proportionally more on their domestic goods even if prices are equal. Second, the ease with which consumers can substitute between goods produced in different countries—the 'trade elasticity'—which governs how relative demand responds to relative prices.

A major reason for this aggregator's wide usage is its tractability, which comes from the fact that the trade elasticity is given by a constant parameter. However, the flipside of this simplicity is that the value of the trade elasticity becomes crucially important for the dynamics of international macro models. In fact, both the sign and size of spillovers from shocks in these models depend on this value. In other strands of the literature, many argue that the trade elasticity can vary with a range of factors (e.g. the level of consumption, the time horizon over which substitution can occur, and income levels). Considering such alternative aggregators, with more parameters and greater flexibility than CES, could allow for richer dynamics and help to alleviate this challenge.

In this paper, we take these alternative aggregators to the workhorse international macro model and assess the implications of how trade aggregation is modelled. How does the precise form of the aggregator influence macro dynamics and the international transmission of shocks?

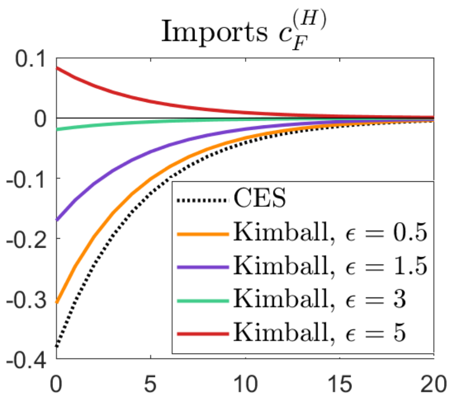
We find that— when linearizing these non-linear relationships—just two sets of parameters are key for macro dynamics: the long-run share of domestic vs. foreign goods in expenditure and the trade elasticity. Since these are precisely the parameters captured in the CES aggregator, our results indicate that the specific aggregator choice has only a limited impact on global macroeconomic model dynamics. This means that, while the precise formulation of the aggregator is irrelevant, the choice of these parameters is important.

In many settings, once you pin down the two sets of sufficient statistics that we identify, it is very hard to overturn the results attained using the standard CES aggregator. This is true in models with only two countries, and in models in which countries are assumed to be perfectly symmetric.

Conversely, by considering models with more than two countries in which countries are asymmetric, we get around this 'aggregator irrelevance'. In these settings, aggregator choice can be important and departing from the CES aggregator can deliver richer dynamics. Indeed, when countries differ in size, or other structural features, there is no longer just a single bilateral trade elasticity to consider, but potentially different elasticities across every pair of goods being traded. As a result, the cross-border transmission of shocks will depend not only on bilateral trade between two countries, but also on their indirect linkages via third countries. While the standard CES aggregator imposes that the elasticity is the same across all pairs of goods, alternative aggregators can allow for realistic differences

in the trade elasticities across different pairs of goods, and therefore qualitatively change the macro dynamics, as is visible in Figure 1.

Figure 1: Impulse Responses of imports of foreign good by home country to a 2% endowment shock in the Home country, asymmetric 3-country case



Note: The Home (H) country is assumed to be a small economy with steady-state endowment equal to 0.5, while the two other countries, Foreign (F) and Rest of World (R), are large (endowment equal to 1). Dotted lines represent the CES responses with $\phi = 1.5$, while each solid line represents responses for different values of the Kimball `curvature' parameter ϵ, with $\sigma = 1.5$. All consumers have symmetric preferences with home bias.

Agrégation entre pays : choix d'agrégateur et dynamiques macroéconomiques

RÉSUMÉ

Nous étudions les implications de l'agrégation entre biens échangeables dans un modèle multi-pays en horizon infini, et si des alternatives à l'agrégateur Armington ont un rôle à jouer dans les modèles typiques de macroéconomie ouverte. Nous montrons analytiquement que les dynamiques de premier ordre du modèle sont entièrement déterminées par un petit nombre de statistiques suffisantes. Au-delà de ces statistiques, le choix précis de la forme fonctionnelle de l'agrégateur du commerce n'a pas d'effet. Ce résultat a les conséquences suivantes. À élasticités de commerce et parts de dépenses à l'état stationnaire données, tout agrégateur homogène de degré 1 est équivalent à l'agrégateur Armington au premier ordre. De même, les agrégateurs homogènes de degré arbitraire sont équivalents à une simple généralisation de l'agrégateur Armington, étant données les élasticités de commerce et parts de dépenses à l'état stationnaire. Dans les modèles avec plus de deux pays, les agrégateurs alternatifs peuvent jouer un rôle en permettant aux élasticités bilatérales de commerce entre différentes paires de pays de prendre des valeurs différentes à l'état stationnaire, ce qui n'est pas possible avec l'agrégateur Armington.

Mots-clés : commerce international, macroéconomie ouverte, agrégateur Armington, élasticité de commerce.

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1 Introduction

Goods trade is a central component of New Open-Economy Macroeconomics (NOEM) models (Corsetti, 2008), playing an important role in the cross-border propagation of macroeconomic shocks. One of the most primitive assumptions in any international macroeconomics model is how domestic and foreign goods are bundled together to form aggregate goods. The structure of this aggregation has implications for how agents' demand responds intratemporally to changes in relative prices. These relative prices can, in turn, influence aggregate wealth and intertemporal consumption-savings decisions and, thus, macroeconomic dynamics. So this aggregation is central to our understanding of many features of the global economy. In this paper, we derive sufficient statistics that summarise the impact of the trade aggregator on the first-order dynamics of these models, and assess the implications of how trade aggregation is modelled for macroeconomic dynamics and the international transmission of shocks.

We show that within the class of aggregators that are homogeneous of degree one, the first-order dynamics of a two-country NOEM model are entirely determined by two types of sufficient statistics: the steady-state consumption expenditure shares and the steady-state elasticity of substitution.[1] The first relate to the share of expenditure on each good, which typically capture the degree of 'home bias' in preferences—the idea that countries tend to spend proportionally more on their domestic goods even if prices are symmetric. The second captures the elasticity of substitution between goods produced in different countries—also known as the 'trade elasticity' or 'Armington elasticity'—governing how relative demand responds to relative prices. Hence, for a given calibration of these steady-state objects, the precise form of the aggregator is irrelevant.

These two sufficient statistics happen to be precisely the quantities that are given parameterically in the Armington (1969) aggregator. The Armington aggregator, which is a Constant Elasticity of Substitution (CES) aggregator, has been widely applied in the NOEM literature, and is the 'go-to' aggregator in multi-country models. A major reason for the Armington aggregator's wide usage is its tractability and elegant closed-form solutions, which are summarised by the two types of parameters that correspond to the sufficient statistics highlighted above. In a two-country model, any aggregator that is homogeneous of degree one is therefore equivalent to the Armington aggregator to first order.

The second part of our contribution extends this result to a broader set-up. We first consider a model with more than two countries. In this case, there is no longer just one single bilateral trade elasticity of substitution, but potentially different elasticities across every pair of country-goods. Our results show that the sufficient statistics for the first-order dynamics of the model now include the steady-state elasticity of substitution across every pair of country-goods. This is because the cross-border transmission of shocks in a multi-country model depends not only

[1] These results are derived analytically from the linearised model, and so they hold exactly at first order. This means that alternative aggregators may have additional effects at higher order, but by definition these effects will be small unless there is a non-linearity in the model, or if we consider large shocks or shocks to higher moments. These extensions are left for future research.

on the bilateral trade between two countries, but also their indirect linkages via third countries.

While the Armington aggregator imposes that the elasticity is the same across all pairs of country-goods, an alternative aggregator can therefore change the first-order dynamics of the model relative to Armington by allowing, in steady state, for different elasticities of substitution across different pairs of goods. This matters for macro modellers as the Armington aggregator remains standard practice even in large-scale, multi-country quantitative models used for policy analysis. For instance, the International Monetary Fund's Global Integrated Monetary and Fiscal model (Laxton, Mursula, Kumhof, and Muir, 2010) features layered CES aggregation of domestic and foreign, consumption and investment, and final and intermediate goods across multiple countries. The use of Armington aggregators in this and other similar cases, by restricting the bilateral Armington elasticity values to a single number for all country pairs, limits the possible dynamics of such large-scale models.

In this context, we also investigate an alternative approach to generating differences in elasticities of substitution across different pairs of goods, while retaining the tractability of the Armington aggregator. Specifically, for each type of good (final goods in our setting), we consider how a nested-CES structure—i.e. a layered set of two-good CES aggregators adding recursively each country-good to the current bundle—compares to alternative aggregators. While allowing for some differences in elasticities across different pairs of goods, we show that this nested-CES structure is still not sufficiently flexible to generically replicate the dynamics under alternative aggregators, as it specifies all bilateral trade elasticities but one, whereas matching all sufficient statistics requires the ability to specify all bilateral elasticities.

Second, we consider the case in which the aggregator is not homogeneous of degree one. The first-order dynamics of the model are then summarised by a broader set of sufficient statistics: the steady state consumption expenditure shares and the steady state bilateral elasticities as before, and the steady state values of ratios related to the degree of homogeneity (if any) of the aggregator. Here, the first-order dynamics will change relative to the Armington aggregator, even in a two-country model, due to the difference in the degree of homogeneity. We propose a simple extension of the Armington aggregator, introducing one new parameter. This generalised Armington aggregator can parsimoniously replicate any aggregator that is homogeneous of arbitrary degree in a two-country setup. As before, in a setup with more than two countries, differences in steady-state bilateral elasticities of substitution can affect the first-order dynamics of the model in a way that the generalised Armington aggregator cannot replicate.

The simplicity and tractability of the Armington aggregator has made it one of the most common aggregator choices in the NOEM literature. However, the flipside of this simplicity is that the value of this single elasticity parameter becomes crucially important for the dynamics of these models when more than two countries are involved, as shown by our sufficient statistics result. As Corsetti, Dedola, and Leduc (2008) demonstrate, both the sign and size of spillovers in NOEM models depend on the trade elasticity in the Armington aggregator. For instance, a low trade elasticity is typically required to match the empirical Backus-Smith-Kollmann correlation—the negative unconditional correlation between real exchange rates and relative consumption.[2] In contrast, micro-evidence of empirically observed patterns of trade substitution point to a high trade elasticity.

The unique elasticity parameter present in the Armington aggregator also plays a role in generating trade comovement across countries. In an example using a three-country set-up, because of the restrictions the unique parameter places on bilateral trade elasticities across country pairs, the Armington aggregator generates conditional responses of trade quantities that counterfactually covary negatively across countries in response to endowment shocks. Within the same model, we show that a Kimball aggregator (following Kimball, 1995) can qualitatively change the dynamics of trade, generating positive cross-country correlation of trade and consumption.[3]

Overall, our analytical results highlight the importance of two types of parameters for trade and, in turn, macroeconomic dynamics in NOEM models. While our work underlines an 'irrelevance' of alternative trade aggregators for macroeconomic dynamics *vis-à-vis* the Armington specification in a two-country setup, there is scope for richer macroeconomic dynamics in more general settings. Moreover, because the sufficient statistics we uncover have clear empirical counterparts, our theoretical findings justify continued focus on the estimation of trade elasticities and shares from micro data (e.g. Freeman, Larch, Theodorakopoulos, and Yotov, 2021).

Related Literature The point of departure for our work is the NOEM literature (e.g Backus, Kehoe, and Kydland, 1992). Within many workhorse multi-country NOEM models, a common assumption is that goods from multiple countries are bundled together to form aggregate consumption using an Armington aggregator (following Armington, 1969) or, in a special case thereof, a Cobb-Douglas aggregator. Within two-country variants of the models, it is widely understood that the two parameters underpinning this aggregation—the elasticity of substitution between traded goods and the degree of openness—are crucial for pinning down the size and, in some circumstances, the sign of cross-country shock transmission (Corsetti et al., 2008). However, while the importance of these Armington parameters is well understood and despite other known limitations of the Armington aggregator, to date no studies have explored how these aggregators can impact shock transmission in NOEM models. Our contribution in this dimension is two-fold. First, applying a generic NOEM model in a two-country setting, we show that Armington aggregation is entirely sufficient for capturing first-order macroeconomic dynamics. Given estimates for steady-state trade elasticities and openness, first-order dynamics are invariant to the aggregation method used. Second, outside of the two-country setting, we show that there is scope to deviate from the first-order dynamics implied by Armington aggregation and that, critical to those deviations, is the calibration of bilateral trade elasticities and openness across each country pair.

[2] Corsetti et al. (2008) show that the Backus-Smith-Kollmann correlation can also be matched with high trade elasticities if shocks are assumed to be persistent.

[3] This finding has parallels with the conclusions of Drozd, Kolbin, and Nosal (2021) who find that changing the household preferences, while keeping a CES bundle of domestic and foreign goods, does not help to resolve the 'trade comovement puzzle'. In contrast, setting a dynamic Armington elasticity can help to resolve the puzzle.

Our paper also contributes to a recent and growing literature searching for sufficient statistics that govern equilibrium outcomes in multi-country and multi-sector models. For example, Arkolakis and Morlacco (2017) analyse the properties of demand functions used in international macroeconomic and trade models as alternatives to the Armington setup. Focusing on the pass-through of marginal costs to prices, they show that, for a general class of demand functions, markups can be written as a function of a single sufficient static, namely the ratio of firm prices to the market price—the 'choke price'. Similarly, and closest in spirit to our contribution, Baqaee and Farhi (2019) investigate the implications of 'Hulten's theorem' (Hulten, 1978) in a multi-sector open-economy setup. They assess the impact of sectoral shocks propagating through global production networks, and derive sufficient statistics in terms of the input-output structure of the economy. In contrast, our work focuses on the dynamics of the workhorse NOEM model. Relative to these papers, a key contribution of our work is to derive results for trade aggregation and macroeconomic dynamics in terms of readily observable sufficient statistics with well-known empirical counterparts. In so doing, our conclusions highlight that recent advances in the estimation of the Armington elasticity, reconciling micro and macro estimates (Feenstra, Luck, Obstfeld, and Russ, 2018), are particularly valuable for the calibration of NOEM models.

Finally, our work brings together insights from the NOEM literature and developments in the trade literature. While the Armington aggregator is commonplace in NOEM models focused on studying the spillovers from macroeconomic shocks, a largely independent literature has put forward a set of alternative functional forms for trade aggregation. Departing from the CES assumption, many of these alternatives allow for variation in the elasticity of substitution in different ways, and this literature has shown that these are key for capturing many empirical facts, for example: variation over time helping to resolve the trade-comovement puzzle (Drozd et al., 2021); variation across firms accounting for variable markups (Kimball, 1995; Gopinath and Itskhoki, 2011); variation with respect to the number of goods being produced (e.g. the QMor and Translog of Bergin and Feenstra, 2000; Feenstra, 2003) capturing the 'pro-competitive' effects of trade (Feenstra, 2018; Arkolakis, Costinot, Donaldson, and Rodríguez-Clare, 2019); and variation in the price-elasticity of demand depending on the consumer's income level, using aggregators that are not homogeneous of degree one (Jung, Simonovska, and Weinberger, 2019). The latter can match pricing-to-market patterns observed in the data, with monopolistically competitive producers setting higher mark-ups and charging higher prices in richer countries. Our contribution is to show in which dimensions and settings will these methods for trade aggregation influence macroeconomic dynamics.

The rest of the paper is organised as follows. Section 2 shows the main features of our generic model. Section 3 contains the core sufficient-statistics result, and Section 4 explores the implications of this in different cases. Section 5 concludes.

2 Model Setup

We begin by setting up a generic multi-country NOEM model. For simplicity and analytical tractability we consider endowment economies, hence abstracting from production and assuming that only final consumption goods are traded across countries.

In this workhorse model, the problem of the representative consumer can be split into an *inter*temporal and an *intra*temporal component. The intertemporal aspect of the household problem is analytically independent of the aggregation structure, and defines aggregate quantities in equilibrium. The intratemporal aspect is aggregator-specific, taking the aggregate choices from the intertemporal problem as given.

For this reason, the results we will present below, about the aggregator choice, are independent of the precise formulation of the intertemporal block of the model. Most notably, the results would continue to hold if we introduced a perfectly competitive production sector, which does not take the demand structure into account, since the intratemporal block remains separate to the production side of the model. Moreover, while we have focused here on the *consumption* aggregator, the same results would hold in more complex models with aggregators used for other types of goods, such as intermediate inputs or investment goods, so long as the optimal composition of these goods between domestic and foreign goods remains an intratemporal choice.

The model has N countries, indexed by $n = 1, 2, ..., N$. Time is discrete and infinite. In each time period t, each country n is endowed with a unique tradable good, denoted by $Y_t^{(n)}$, which takes strictly positive values. Variation in these country-specific endowments is the sole source of uncertainty in our model. The endowments are subject to stochastic mean-zero disturbances from period to period, which result in fluctuations around their mean value, denoted by $\overline{Y}^{(n)}$. Hence the steady state of the model is defined as the deterministic equilibrium with $Y_t^{(n)} = \overline{Y}^{(n)}$ for all t.

Intertemporal Problem. The representative consumer in country n has additively separable preferences over time:

$$U_t^{(n)} = \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^\tau u\left(C_{t+\tau}^{(n)}\right) \right]$$

where $C_t^{(n)}$ denotes aggregate consumption in period t; $u : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable, strictly increasing and strictly concave function, with $\lim_{C\to 0} u'(C) = \infty$; and $\beta \in (0, 1)$ is the discount factor.

Let $P_t^{(n)}$ denote the price of a unit of aggregate consumption in country n and $P_{i,t}^{(n)}$ the price of a unit of the country-i good in country n. The intertemporal budget constraint of the country-n representative consumer is:

$$\sum_{\tau=0}^{\infty} \left(P_{t+\tau}^{(n)} C_{t+\tau}^{(n)} - P_{n,t+\tau}^{(n)} Y_{t+\tau}^{(n)} \right) \leq 0.$$

Without loss of generality, we will consider complete international capital markets. We obtain the equilibrium risk-sharing condition by equalising the optimality conditions for the representative households in two countries, n and n':

$$RER_{t+\tau}^{(n,n')} = \kappa^{(n,n')} \frac{u'\left(C_{t+\tau}^{(n')}\right)}{u'\left(C_{t+\tau}^{(n)}\right)} \qquad \forall\, n' \neq n$$

where $RER_t^{(n,n')} \equiv P_t^{(n')}/P_t^{(n)}$ denotes the real exchange rate of country n *vis-à-vis* country n', defined such that an increase in its value represents a depreciation for country n.

Assumption 1: In steady state, there is bilateral balanced trade between every pair of countries.

The risk-sharing constant, $\kappa^{(n,n')}$, ensures that the risk-sharing condition is satisfied at steady state under Assumption 1. Specifically, $\kappa^{(n,n')} = 1$ when countries are symmetric at steady state, but it will differ from unity when there are steady-state asymmetries across countries.[4]

In equilibrium, the intertemporal optimisation of representative consumers in each country, and the risk-sharing condition, will pin down the sequence of $C_t^{(n)}$ and $RER_t^{(n,n')}$ given the endowment processes.

Intratemporal Problem. The aggregate consumption of households in country n is formed of goods produced in all N countries, according to the aggregator function $f : \mathbb{R}_+^N \to \mathbb{R}$, such that:

$$C_t^{(n)} \equiv f\left(\mathbf{c}_t^{(n)}\right) \tag{1}$$

where $\mathbf{c}_t^{(n)} = \left[c_{1,t}^{(n)}, c_{2,t}^{(n)}, ..., c_{N,t}^{(n)}\right]'$ denotes the $N \times 1$ vector of consumption levels, with $c_{i,t}^{(n)}$ denoting the representative country-n household's consumption of goods from country i.

The intratemporal problem of the representative household at time t involves minimising total expenditure, taking as given the level of aggregate consumption from the intertemporal optimisation, $C_t^{(n)}$, and the prices, $P_{i,t}^{(n)}$.

Assumption 2: The law of one price (LOOP) holds, such that $P_{i,t}^{(n)} = P_{i,t}$ for all n.

Using Assumption 2, the intratemporal problem can be written as:

$$\min_{\mathbf{c}_t^{(n)}} \sum_{i=1}^{N} P_{i,t} c_{i,t}^{(n)} \quad \text{subject to} \quad C_t^{(n)} = f\left(\mathbf{c}_t^{(n)}\right).$$

Assumption 3: The function f is continuous, twice differentiable and strictly quasi-concave.

Under Assumption 3, the solution to the intratemporal problem exists, is unique, and is defined

[4]As long as Assumption 1 is satisfied, our results will hold under financial autarky or other forms of incomplete markets.

by the first-order conditions:

$$P_{i,t} = \lambda_t f_{i,t}^{(n)} \quad \forall\, i = 1, \ldots N$$

where λ_t is the Lagrange multiplier on the constraint and $f_{i,t}^{(n)} \equiv \partial f(\mathbf{c}_t^{(n)})/\partial c_{i,t}^{(n)}$. To remove the Lagrange multiplier, these N optimality conditions can be written as $(N-1)$ relative demand functions:

$$\frac{f_{i,t}^{(n)}}{f_{N,t}^{(n)}} = \frac{p_{i,t}^{(n)}}{p_{N,t}^{(n)}} \quad \text{for } i = 1, 2, ..., N-1. \tag{2}$$

where we have also defined the relative price as $p_{i,t}^{(n)} \equiv P_{i,t}/P_t^{(n)}$.

The aggregate consumer price index $P_t^{(n)}$ can be defined simply by:

$$\begin{aligned} P_t^{(n)} C_t^{(n)} &= \sum_{i=1}^{N} P_{i,t} c_{i,t}^{(n)} \\ C_t^{(n)} &= \sum_{i=1}^{N} p_{i,t}^{(n)} c_{i,t}^{(n)} \end{aligned} \tag{3}$$

World equilibrium in goods markets is given by:

$$Y_t^{(n)} \geq \sum_{i=1}^{N} c_{n,t}^{(i)} \quad \text{for } n = 1, 2, ..., N. \tag{4}$$

In equilibrium, the intratemporal optimisation in each country, and the goods market clearing conditions, define the trade quantities $\mathbf{c}_t^{(n)}$ and relative prices $\mathbf{p}_t^{(n)}$, given aggregate variable definitions from the intertemporal problem.

3 Sufficient Statistics for the Aggregator

The key question of this paper is how the specific choice of functional form for f affects the model's equilibrium macroeconomic dynamics. Our main result is summarised by the following theorem:

Theorem 1 *Under Assumptions 1-3, the effect of the aggregator function on the first-order dynamics of this model is captured entirely by the following sufficient statistics, where overlines represent the steady-state values of variables and functions thereof:*

(i) the elasticities of substitution between each pair of goods:

$$\overline{\Phi}_{i,j}^{(n)} \equiv \frac{\partial \ln\left(\overline{c}_i^{(n)}/\overline{c}_j^{(n)}\right)}{\partial \ln\left(\overline{f}_j^{(n)}/\overline{f}_i^{(n)}\right)} \quad \textit{for } i,\ j = 1, 2, ..., N,\ i \neq j$$

(ii) the share of consumption expenditure for each good:

$$\overline{\alpha}_i^{(n)} \equiv \frac{\overline{p}_i^{(n)} \overline{c}_i^{(n)}}{\overline{C}^{(n)}} \quad \text{for } i = 1, 2, ..., N,$$

(iii) the ratio $\overline{\mathcal{H}}^{(n)}$, defined as:

$$\overline{\mathcal{H}}^{(n)} \equiv \mathcal{H}(\overline{\mathbf{c}}^{(n)}) = \frac{\sum_{i=1}^{N} \overline{f}_i^{(n)} \overline{c}_i^{(n)}}{f(\overline{\mathbf{c}}^{(n)})}$$

(iv) the ratios $\overline{\mathcal{H}}_i^{(n)}$ for each good, defined as:

$$\overline{\mathcal{H}}_i^{(n)} \equiv \mathcal{H}_i(\overline{\mathbf{c}}^{(n)}) = \frac{\sum_{k=1}^{N} \overline{f}_{ik}^{(n)} \overline{c}_k^{(n)}}{f_i(\overline{\mathbf{c}}^{(n)})} \quad \text{for } i = 1, 2, ..., N,$$

for each country $n = 1, 2, ..., N$.

Proof: First, notice that equations (1)-(4) are the only model equations affected by the aggregator function and the consumption levels $\mathbf{c}_t^{(n)}$. The rest of the model equations are independent of the aggregator by definition. We therefore prove the theorem by showing that the first-order approximation of these four equations only depends on the aggregator function, f, through the steady-state quantities described above. Full derivations are provided in Appendix A. □

The elasticities of substitution, consumption expenditure shares and the ratios $\mathcal{H}(.)$ and $\mathcal{H}_i(.)$ are generically functions of the variables of the model and therefore can vary dynamically. However, Theorem 1 states that the dynamics of the model at first order depend only on the *steady-state* values of these objects.

Before unpacking the implications of this theorem, it is useful to say a few words on $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$. While it is natural to think of an aggregator as being defined by the elasticities of substitution and the consumption shares across goods, the ratios $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$ are less familiar. To better understand what these two statistics are, it is useful to consider a specific class of aggregators: homogeneous functions. The result for this group is summarised by the following corollary to the theorem:

Corollary 1 *If the aggregator function, f, is homogeneous of degree h, the first-order dynamics of the model are captured by the following sufficient statistics: $\overline{\Phi}_{i,j}^{(n)}$, $\overline{\alpha}_i^{(n)}$ for $i, j = 1, 2, ..., N$, $i \neq j$, $n = 1, 2, ..., N$, as defined above, and h.*

Proof: Recall, first, that if a function is homogeneous of degree h, then the partial derivatives of that function are homogeneous of degree $(h-1)$. Then, by Euler's theorem, if the function f is homogeneous of degree h, then $\overline{\mathcal{H}}^{(n)} = h$ and $\overline{\mathcal{H}}_i^{(n)} = (h-1)$. Hence, h becomes the sufficient statistic to replace $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$. Full derivations are provided in Appendix B. □

When comparing across homogeneous aggregators, Corollary 1 has implications that are summarised in the following corollary:

Corollary 2 *All aggregators that are homogeneous of the same degree will imply the same first-order dynamics, for given* $\overline{\Phi}_{i,j}^{(n)}$ *and* $\overline{\alpha}_i^{(n)}$ *for* $i,\ j = 1, 2, ..., N,\ \ i \neq j,\ \ n = 1, 2, ..., N,$ *as defined above.*

Proof: This follows directly from Corollary 1. When comparing across aggregators with the same h, then the sufficient statistics collapse to just the elasticities and expenditure shares. □

To explore the implications of Theorem 1 and Corollaries 1 and 2, the following section considers a few separate cases.

4 Implications of the Theorem

4.1 Homothetic Preferences

One of the basic assumptions of most economic models is that preferences are homothetic. A homothetic function is a monotonic transformation of a function that is homogeneous of degree 1, henceforth referred to as HOD(1). Therefore, if the utility function, $u(C_t^{(n)})$, is a monotonic increasing function of aggregate consumption, then utility is homothetic with respect to $\mathbf{c}_t^{(n)}$ if the aggregator function f is HOD(1).

Corollary 2 implies that, within the class of HOD(1) aggregators, the sufficient statistics for the first-order dynamics of the model are just the steady-state elasticities of substitution and expenditure shares. Since this class includes the Armington aggregator, this means that any alternative HOD(1) aggregator, with the same steady-state elasticities of substitution and expenditure shares, will be equivalent to the Armington aggregator.

In this subsection, we unpack these implications by comparing the Armington aggregator to the Kimball (1995) aggregator, an alternative HOD(1) functional form. Our exploration proceeds in three steps: (i) we consider the two-country case, (ii) we extend our analysis to more than two countries, and (iii) we compare these results to a nested-CES framework.

4.1.1 Case 1: Two Countries

If $N = 2$*, all aggregators that are HOD(1) are equivalent at first order to the Armington aggregator with the same steady-state elasticity and home bias.*

Armington Aggregator. The two-country Armington aggregator is given by:

$$C_t^{(n)} \equiv f(c_{1,t}^{(n)}, c_{2,t}^{(n)}) = \left(a_1^{(n)\,\frac{1}{\phi}} c_{1,t}^{(n)\,\frac{\phi-1}{\phi}} + \left(1 - a_1^{(n)}\right)^{\frac{1}{\phi}} c_{2,t}^{(n)\,\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}} \quad \text{for } n = 1, 2$$

and yields the familiar relative demand functions:

$$\frac{c_{1,t}^{(n)}}{c_{2,t}^{(n)}} = \frac{a_1^{(n)}}{1 - a_1^{(n)}} \left(\frac{p_{2,t}^{(n)}}{p_{1,t}^{(n)}} \right)^{\phi} \quad \text{for } n = 1, 2$$

where ϕ is the constant elasticity of substitution between the only two goods, and $a_1^{(1)}$ is the degree of home bias in country n, which maps into the steady-state consumption shares, $\overline{\alpha}_1^{(1)}$ and $\overline{\alpha}_2^{(1)} = (1 - \overline{\alpha}_1^{(1)})$.[5]

The first-order approximation of these equations is given by:

$$\widetilde{C}_t^{(n)} = \overline{\alpha}_1^{(n)} \widetilde{c}_{1,t}^{(n)} + (1 - \overline{\alpha}_1^{(n)}) \widetilde{c}_{2,t}^{(n)} \tag{5}$$

$$\widetilde{c}_{1,t}^{(n)} - \widetilde{c}_{2,t}^{(n)} = \phi \left(\widetilde{p}_{2,t}^{(n)} - \widetilde{p}_{1,t}^{(n)} \right) \tag{6}$$

for $n = 1, 2$, where $\widetilde{x}_t$ is the percentage deviation of variable x from its steady state $\bar{x}$. These two equations illustrate how the two parameters of the Armington aggregator enter the linearised model.

Corollary 2 tells us that any HOD(1) aggregator across two goods can be mapped into an equivalent Armington aggregator, with ϕ set to match the same *steady-state* elasticity of substitution, and $a_1^{(n)}$ set to match the same *steady-state* consumption shares.

To illustrate this property, we compare these linearised equations under CES to the Kimball (1995) aggregator—an alternative HOD(1) specification.

Kimball Aggregator. Consider Kimball (1995)'s aggregator, where aggregate consumption $C_t^{(n)}$ is implicitly defined by:

$$1 = b_1^{(n)} \Upsilon \left(\frac{c_{1,t}^{(n)}}{b_1^{(n)} C_t^{(n)}} \right) + b_2^{(n)} \Upsilon \left(\frac{c_{2,t}^{(n)}}{b_2^{(n)} C_t^{(n)}} \right) \quad \text{for } n = 1, 2 \tag{7}$$

where $b_2^{(n)} \equiv (1 - b_1^{(n)})$, and $\Upsilon(.)$ is such that $\Upsilon(1) = 1$, $\Upsilon'(.) > 0$ and $\Upsilon''(.) > 0$. It can be seen from this implicit definition of $C_t^{(n)}$ that aggregate consumption is HOD(1) in consumption of country-specific goods: increasing both $c_{1,t}^{(n)}$ and $c_{2,t}^{(n)}$ by the same factor would require $C_t^{(n)}$ to increase by the same factor for the implicit function to continue to hold.

[5] In a symmetric steady state, in which the prices of the two goods are equal, then $\overline{\alpha}_1^{(1)} = a_1^{(1)}$, but outside of symmetry this mapping will depend on the steady-state relative prices, with $\overline{\alpha}_1^{(1)} = a_1^{(1)} (\overline{p}_1^{(1)} / \overline{P}^{(1)})^{1-\phi}$.

We follow Klenow and Willis (2016) and specify the function $\Upsilon(.)$ as:[6]

$$\Upsilon(x) = 1 + (\sigma - 1)\exp(\epsilon^{-1})\epsilon^{\frac{\sigma}{\epsilon}-1}\left(\Gamma\left(\frac{\sigma}{\epsilon}, \frac{1}{\epsilon}\right) - \Gamma\left(\frac{\sigma}{\epsilon}, \frac{x^{\frac{\epsilon}{\sigma}}}{\epsilon}\right)\right)$$

where:

$$\Gamma(u, z) = \int_z^{+\infty} s^{u-1}\exp(-s)\,\mathrm{d}s$$

This specification of $\Upsilon(.)$ yields the following derivative:

$$\Upsilon'(x) = \frac{\sigma - 1}{\sigma}\exp\left\{\frac{1 - x^{\frac{\epsilon}{\sigma}}}{\epsilon}\right\}$$

This aggregator is defined by three parameters: σ, ϵ and $b_1^{(n)}$ for each country. $b_1^{(n)}$ is a familiar home-bias parameter, which maps into consumption shares, while σ and ϵ pin down the elasticity of substitution.

To see this, consider the relative demand functions from the household's intratemporal problem:

$$\frac{p_{1,t}^{(n)}}{p_{2,t}^{(n)}} = \frac{\Upsilon'\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)}{\Upsilon'\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)}C_t^{(n)}}\right)} = \frac{\exp\left\{\frac{1}{\epsilon}\left(1 - \left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}}{\exp\left\{\frac{1}{\epsilon}\left(1 - \left(\frac{c_{2,t}^{(n)}}{b_2^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}} \quad \text{for } n = 1, 2 \tag{8}$$

From this, we can define the consumption shares and elasticity of substitution:

$$\alpha_{1,t}^{(n)} = \frac{p_{1,t}^{(n)}c_{1,t}^{(n)}}{p_{1,t}^{(n)}c_{1,t}^{(n)} + p_{2,t}^{(n)}c_{2,t}^{(n)}} = \frac{\Upsilon'\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)c_{1,t}^{(n)}}{\Upsilon'\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)c_{1,t}^{(n)} + \Upsilon'\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)}C_t^{(n)}}\right)c_{2,t}^{(n)}} \tag{9}$$

$$\Phi_{1,2,t}^{(n)} = \sigma\left(1 + \frac{\alpha_{1,t}^{(n)}}{\alpha_{2,t}^{(n)}}\right)\left[\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}} + \frac{\alpha_{1,t}^{(n)}}{\alpha_{2,t}^{(n)}}\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right]^{-1} \tag{10}$$

for $n = 1, 2$.[7]

Equation (10) illustrates the key property of Kimball preferences: the elasticity of substitution depends on the relative consumption levels. Notice that as $\epsilon \to 0$, $\Phi_{1,2,t}^{(n)} \to \sigma$, implying that Kimball nests CES, with elasticity σ, as a limit case.

To further explore the properties of the Kimball aggregator, Figure 1 plots the relative demand function, given by equation (8), and corresponding elasticities from equation (10), for different

[6] There are multiple formulations of the Kimball (1995) aggregator, with different specifications of $\Upsilon(.)$. For example, Lindé and Trabandt (2018) use a Dotsey and King (2005) specification in their closed-economy analysis. But the specific choice of functional form is irrelevant for our results.

[7] Full derivations are provided in Appendix C.

values of ϵ. To form this plot, we calibrate the three remaining aggregator parameters: $\sigma = 1.5$, $b_1^{(n)} = 0.8$ and $b_2^{(n)} = 0.2$. First, notice that when $p_1^{(n)}/p_2^{(n)} = 1$, we have $c_1^{(n)}/c_2^{(n)} = b_1^{(n)}/b_2^{(n)} = 4$ and $\Phi_{1,2}^{(n)} = \sigma = 1.5$ independently of ϵ. This implies that Kimball is also equivalent to CES at the point of symmetry across good types, with $\sigma = \phi$ and $b_1^{(n)} = a_1^{(n)}$.[8]

More generally, ϵ controls the curvature of the demand function. In the limiting case of CES preferences, as $\epsilon \to 0$, shown in the black dotted lines, the relative demand function is convex and the elasticity of substitution is constant at $\sigma = 1.5$. As ϵ increases, the relative demand curve becomes less convex, and the elasticity of substitution varies with the relative consumption levels. For $\epsilon = \sigma$, the relative demand curve is approximately linear. When $\epsilon > \sigma$, the curve is concave. When this is the case, the concave relative demand curves imply finite 'choke prices', above which demand for the relatively more expensive good is 0.

Consider, for example, the concave relative demand at $\epsilon = 5$ in panel (a) of Figure 1. Here, as the price of good 1 relative to good 2 rises above 1, the concavity of the curve means that relative demand for good 1 falls more than in the CES case. In contrast, when the relative price of good 1 falls below 1, the concavity of the curve means that the relative demand for good 1 rises less rapidly than it does under CES.

We can explain this equivalently in terms of the elasticity of substitution, shown in panel (b) of Figure 1. When consumption of good 1 is low relative to good 2, then the elasticity of substitution rises, and a decrease in the relative price of good 1 leads to a larger substitution towards good 1. Conversely, when the consumption of good 1 is high relative to good 2, then the elasticity of substitution falls, and a decrease in the relative price of good 1 leads to a smaller substitution towards good 1.

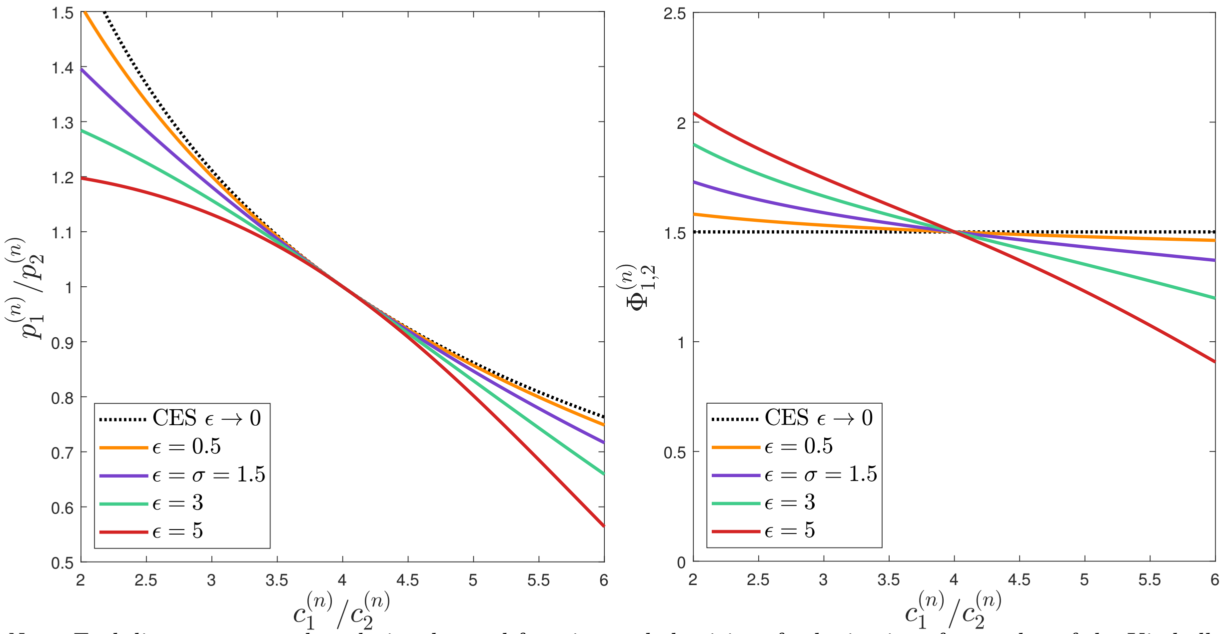
This relative-demand curvature allows for the elasticity of substitution to vary over time, as the economy is hit by exogenous shocks, as illustrated by the time subscripts in equation (10). This leads to what Klenow and Willis (2016) refer to as "a smoothed version of a kinked demand curve": if a shock drives the relative price of a good up, the elasticity of substitution increases, such that demand declines more than the CES case, while if a shock drives the relative price down, the elasticity decreases, such that demand increases less than the CES case.

Comparing Armington and Kimball. Despite these additional mechanisms in the Kimball aggregator, the application of Corollary 2 to this case tells us that, at first order, Kimball is equivalent to the Armington aggregator. To see why, we take the first-order approximation of the implicit definition of aggregate consumption, equation (7), and the relative demand function,

[8]This point is explored more in Baqaee, Farhi, and Sangani (2021), who highlight the importance of firm heterogeneity when using the Kimball aggregator to aggregate across monopolistically differentiated goods.

Figure 1: Kimball (1995) aggregator

(a) Relative demand functions (b) Elasticity of substitution



Note: Each line represents the relative demand function and elasticity of substitutions for a value of the Kimball 'curvature' parameter ϵ, plotted with $\sigma = 1.5$ and $b_1^{(n)} = 0.8$.

equation (8):

$$\widetilde{C}_t^{(n)} = \overline{\alpha}_1^{(n)} \widetilde{c}_{1,t}^{(n)} + (1 - \overline{\alpha}_1^{(n)}) \widetilde{c}_{2,t}^{(n)}$$
$$\widetilde{c}_{1,t}^{(n)} - \widetilde{c}_{2,t}^{(n)} = \overline{\Phi}_{1,2}^{(n)} \left(\widetilde{p}_{2,t}^{(n)} - \widetilde{p}_{1,t}^{(n)} \right)$$

for $n = 1, 2$, where $\overline{\alpha}_1^{(n)}$ and $\overline{\Phi}_{1,2}^{(n)}$ are the steady-state values of the consumption share and elasticity of substitution defined in equations (9) and (10).[9]

From these equations we see that, even under Kimball, the linearised equations only depend on the *steady-state* consumption shares and the *steady-state* elasticity of substitution. The parameters of the Kimball aggregator, including the curvature parameter ϵ, only matter insofar as they pin down these two steady-state values. Importantly, then, despite the fact that $\epsilon > 0$ allowed for the elasticity of substitution to vary dynamically, as described above, these dynamics do not enter the linearised model equations.

Thus, for a given value of the Kimball parameters, we can set the Armington parameters, $a_1^{(n)}$ to match the same $\overline{\alpha}_1^{(n)}$, and $\phi = \overline{\Phi}_{1,2}^{(n)}$, and we see immediately that these linearised equations are exactly equivalent to the linearised equations under CES, equations (5) and (6). In other words, the first-order dynamics of the two-country model with the Kimball aggregator are equal to those of a CES specification, for given steady-state consumption shares and steady-state elasticity of

[9]These expressions can be derived by applying the generic formulas in the proof of Theorem 1 in Appendix A.

substitution.

4.1.2 Case 2: $N > 2$ Countries

If $N > 2$, then the specific form of the aggregator is relevant only to the extent that the bilateral elasticities of substitution across different pairs of goods are different in steady state.

Armington Aggregator. We can define the N-good Armington aggregator in country-n as:

$$C_t^{(n)} = f(c_{1,t}^{(n)}, \dots c_{N,t}^{(n)}) = \left(\sum_{i=1}^{N} a_i^{(n)\frac{1}{\phi}} c_{i,t}^{(n)\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}} \qquad \text{for } n \in [1, N]$$

where $\sum_{i=1}^{N} a_i^{(n)} = 1$.

The relative demand functions are given by:

$$\frac{c_{i,t}^{(n)}}{c_{N,t}^{(n)}} = \frac{a_i^{(n)}}{a_N^{(n)}} \left(\frac{p_{N,t}^{(n)}}{p_{i,t}^{(n)}} \right)^{\phi} \qquad \text{for } i \in [1, N-1],\ n \in [1, N]$$

This leads to the following linearised equations:

$$\widetilde{C}_t^{(n)} = \sum_{i=1}^{N} \overline{\alpha}_i^{(n)} \widetilde{c}_{i,t}^{(n)}$$

$$\widetilde{c}_{i,t}^{(n)} - \widetilde{c}_{N,t}^{(n)} = \phi \left(\widetilde{p}_{N,t}^{(n)} - \widetilde{p}_{i,t}^{(n)} \right) \qquad \text{for } i \in [1, N-1]$$

for $n \in [1, N]$.

The pair-wise elasticities of substitution between any two goods are given by the same parameter, ϕ, by definition of the single Armington aggregator. To see how this property affects the comparison with more general aggregators, we go back to the example of the Kimball aggregator considered above.

Kimball Aggregator. In each country n, the implicit definition of the N-good Kimball aggregator is now:

$$1 = \sum_{i=1}^{N} b_i^{(n)} \Upsilon \left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}} \right) \qquad \text{for } n \in [1, N]$$

where $\sum_{i=1}^{N} b_i^{(n)} = 1$ and the function $\Upsilon(.)$ is defined as in Section 4.1.1.

The resulting relative demand functions are then:

$$\frac{p_{i,t}^{(n)}}{p_{N,t}^{(n)}} = \frac{\Upsilon'\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)}C_t^{(n)}}\right)}{\Upsilon'\left(\frac{c_{N,t}^{(n)}}{b_N^{(n)}C_t^{(n)}}\right)} = \frac{\exp\left\{\frac{1}{\epsilon}\left(1-\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}}{\exp\left\{\frac{1}{\epsilon}\left(1-\left(\frac{c_{N,t}^{(n)}}{b_N^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}} \quad \text{for } i \in [1, N-1],\ n \in [1, N]$$

As for the two-country case, we can compute the consumption shares and the bilateral elasticities of substitution:

$$\alpha_{i,t}^{(n)} = \frac{p_{i,t}^{(n)} c_{i,t}^{(n)}}{\sum_{j=1}^{N} p_{j,t}^{(n)} c_{j,t}^{(n)}} = \frac{c_{i,t}^{(n)}\Upsilon'\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)}C_t^{(n)}}\right)}{\sum_{j=1}^{N} c_{j,t}^{(n)}\Upsilon'\left(\frac{c_{j,t}^{(n)}}{b_j^{(n)}C_t^{(n)}}\right)} \quad \text{for } i \in [1, N]$$

$$\Phi_{i,j,t}^{(n)} = \sigma\left(1 + \frac{\alpha_{i,t}^{(n)}}{\alpha_{j,t}^{(n)}}\right)\left[\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}} + \frac{\alpha_{i,t}^{(n)}}{\alpha_{j,t}^{(n)}}\left(\frac{c_{j,t}^{(n)}}{b_j^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right]^{-1} \quad \text{for } i, j \in [1, N],\ i \neq j \tag{11}$$

for $n \in [1, N]$.[10]

As before, the elasticity of substitution depends on the relative consumption levels. As well as allowing the elasticity to vary over time, we see that the elasticity can be different for different pairs of goods, depending on the asymmetries between countries. From equation (11), it is easy to see that the elasticities between two pairs of goods, $\{i, j\}$ and $\{i, l\}$, will be equal if and only if one of three conditions holds: (i) $\epsilon = 0$, in which case we are back to the CES aggregator with $\Phi_{i,j,t}^{(n)} = \sigma$; (ii) $\alpha_{i,t}^{(n)} = 0$, implying that good i is not consumed at all; or, most importantly, (iii) $b_j^{(n)} = b_l^{(n)}$ and $c_{j,t}^{(n)} = c_{l,t}^{(n)}$, such that consumption shares are equal across goods. Ignoring the trivial cases (i) and (ii), we therefore see that the Kimball aggregator implies that the elasticities across different pairs of goods will be different, unless there is perfect symmetry across all countries.

Comparing Armington and Kimball. To see how these differences in elasticities across different country-pairs affect the dynamics of the model, we again take the first-order approxi-

[10] Full derivations are provided in Appendix C.

mation of the aggregator and relative demand functions under Kimball:

$$\begin{aligned}
\widetilde{C}_t^{(n)} &= \sum_{i=1}^{N} \overline{\alpha}_i^{(n)} \widetilde{c}_{i,t}^{(n)} \\
\widetilde{p}_{i,t}^{(n)} - \widetilde{p}_{N,t}^{(n)} &= \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}_k^{(n)} \sum_{l=1, l\neq k}^{N} \Bigg[\alpha_k^{(n)} \left(\left(\overline{\Phi}_{Nl}^{(n)}\right)^{-1} - \left(\overline{\Phi}_{il}^{(n)}\right)^{-1} \right) + \alpha_l^{(n)} \left(\left(\overline{\Phi}_{ik}^{(n)}\right)^{-1} - \left(\overline{\Phi}_{Nk}^{(n)}\right)^{-1} \right) \\
&\quad + \frac{\alpha_k^{(n)} \alpha_l^{(n)}}{\alpha_N^{(n)}} \left(\left(\overline{\Phi}_{Nl}^{(n)}\right)^{-1} - \left(\overline{\Phi}_{Nk}^{(n)}\right)^{-1} \right) + \frac{\alpha_k^{(n)} \alpha_l^{(n)}}{\alpha_i^{(n)}} \left(\left(\overline{\Phi}_{ik}^{(n)}\right)^{-1} - \left(\overline{\Phi}_{il}^{(n)}\right)^{-1} \right) \Bigg] \\
&\qquad \text{for } i \in [1, N-1]
\end{aligned}$$

for $n \in [1, N]$.[11]

We can see that the presence of the additional countries creates additional terms in the relative demand function, capturing the potential indirect substitution between goods i and N via goods k, l. Importantly, when we have perfect symmetry across countries in steady state, such that $\overline{\Phi}_{i,j}^{(n)} = \overline{\Phi}_{i,l}^{(n)}$ for all i, j and l, then these additional terms disappear from all relative demand functions.[12] This is why these terms were absent for the Armington aggregator. In this symmetric case, therefore, we can again replicate the first-order dynamics from the Kimball aggregator using an Armington aggregator by matching the steady-state consumption shares, and setting ϕ to match this common elasticity of substitution.

However, if we allow for steady-state asymmetries across countries, then these additional terms will create first-order effects that cannot be captured by an Armington aggregator. Notice that it is again only the steady-state values of the elasticities that enter the linearised equations, and not any dynamic variation in the elasticity. Nonetheless, the Kimball aggregator allows us to map steady-state asymmetries in, say, endowments, into differences in elasticities of substitution, which then impacts the dynamics of the model.

Numerical Exercise with $N = 3$. To illustrate these effects of using the Kimball aggregator, we consider a three-country version of our model. We label the countries $n = \{H, F, R\}$ and consider our results from the perspective of the Home country, H. For this stylised exercise, we set the discount factor $\beta = 0.99$, and assume the instantaneous utility function $u(.)$ has the constant relative risk aversion form, with a coefficient of relative risk aversion of 2.

Recall that, under symmetry, Armington and Kimball aggregators can be equivalent, regardless of the value of the Kimball ϵ parameter, with the right choice of the remaining parameters. Making use of this, we set the parameters of the Armington and Kimball aggregators so that they are equivalent in the symmetric steady state. We then keep these parameters fixed in the

[11] As before, these expressions are derived within the proof of Theorem 1 in Appendix A for a generic aggregator.

[12] We use the convention that $\Phi_{i,i} = 0$ for all i, so that in the symmetric case, the terms of the equation where such same-good elasticities appear will not simplify away despite the symmetry, and the linearised equation remains valid.

asymmetric setting, allowing the elasticities and expenditure shares to vary, and compare them across different values of ϵ.

In particular, we set $a_i^{(i)} = b_i^{(i)} = 0.7$ and $a_j^{(i)} = b_j^{(i)} = 0.15$ for all $i, j \in \{H, F, R\}$, $j \neq i$. This implies that in the symmetric steady states, the domestic expenditure share is 70%, and the remaining expenditure share is split equally across the two foreign countries. Similarly, we set $\phi = \sigma = 1.5$, so that, under symmetry, $\overline{\Phi}_{i,j}^{(n)} = 1.5$ for all $i, j, n \in \{H, F, R\}$, $j \neq i$.

To illustrate the role of the Kimball aggregator, we then depart from symmetry by assuming the steady-state endowment of country H is smaller than the endowment of F and R. Normalising these values, we set $\overline{Y}^{(F)} = \overline{Y}^{(R)} = 1$ and $\overline{Y}^{(H)} = 0.5$. Table 1 shows the implied values of the steady-state objects of interest, across the different values of ϵ, where the $\epsilon = 0$ column corresponds to the Armington case.

Across all values of ϵ, this reduction in the supply of country-H goods increases its steady-state relative price above 1. Given that the Home good's relative international price is now higher, Home agents consume a higher share of their domestic good. Reflecting this high relative consumption and in line with the 'kinked demand curve' mechanism explained with reference to Figure 1, the Home consumer's elasticity of substitution is declining in ϵ and smaller than in the Armington case.

Table 1: Steady-State Expenditure Shares and Elasticities of Substitution Under Asymmetry

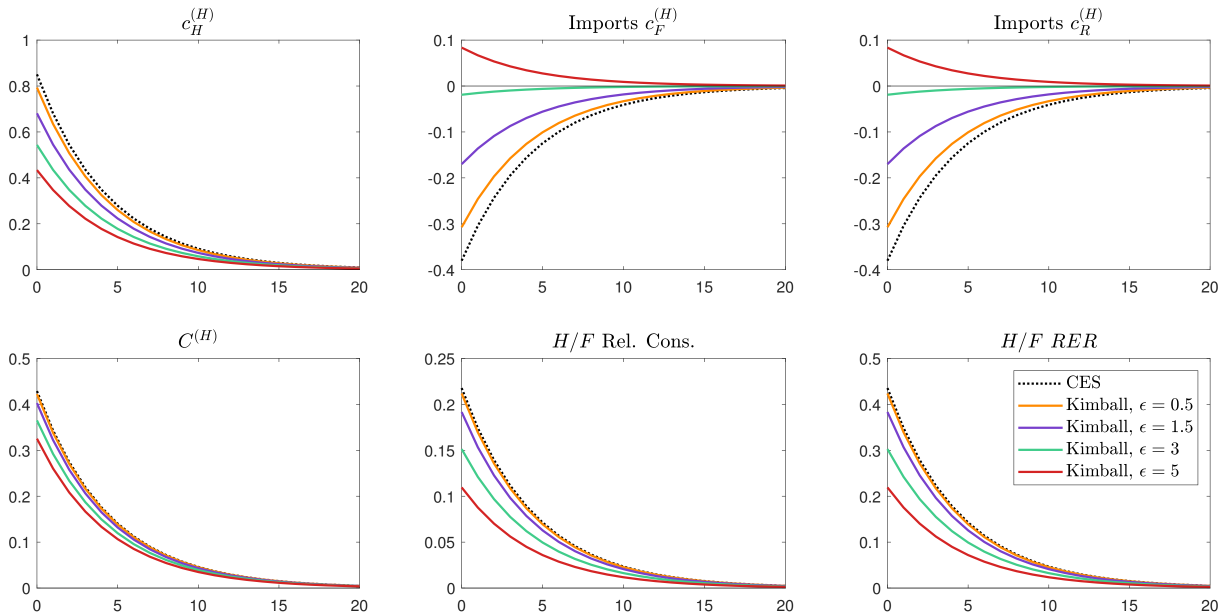
	Symmetry	Asymmetry				
		$\epsilon = 0$	$\epsilon = 0.5$	$\epsilon = 1.5$	$\epsilon = 3$	$\epsilon = 5$
$\bar{p}_H^{(H)}/\bar{p}_F^{(H)}$	1	1.48	1.46	1.41	1.35	1.28
$\bar{c}_H^{(H)}/\bar{c}_F^{(H)}$	1	3.84	3.94	4.12	4.31	4.45
$\bar{\alpha}_H^{(H)}$	70	65.74	66.32	67.31	68.29	68.99
$\bar{\alpha}_F^{(H)}$	15	17.13	16.84	16.35	15.86	15.51
$\bar{\Phi}_{H,F}^{(H)}$	1.5	1.50	1.37	1.17	0.99	0.86
$\bar{\Phi}_{F,R}^{(H)}$	1.5	1.50	1.33	1.09	0.89	0.76

Note: Due to the symmetry between F and R, $\bar{p}_F^{(H)} = \bar{p}_R^{(H)}$, $\bar{\alpha}_F^{(H)} \equiv \bar{\alpha}_R^{(H)}$ and $\bar{\Phi}_{H,F}^{(H)} \equiv \bar{\Phi}_{H,R}^{(H)}$.

Figure 2 shows impulse response functions to a 2% increase in Home endowment. After a positive endowment shock, the aggregate consumption in the home country, $C^{(H)}$, always increases, more so with CES or more convex Kimball preferences ($\epsilon < \phi$), while the Home real exchange rate depreciates as the relative price of the Home good decreases.

More interestingly, the Home consumer's consumption responses for goods F and R change qualitatively with the curvature of Kimball preferences. The intuition behind this is as follows. When their relative demand function is more concave (ϵ larger), given that they already consume a large quantity of Home good, the steady-state Home elasticity of substitution for H and F goods $\overline{\Phi}_{H,F}^{(H)}$ is relatively low. So the Home consumer flocks towards the cheaper Home good more slowly. They will rather use their additional endowment to consume more of the F and

Figure 2: IRFs to a 2% endowment shock in the Home country, asymmetric 3-country case



Note: The Home (H) country is assumed to be a small economy with steady-state endowment equal to 0.5, while the two other countries, (F) and (R), are large (endowment equal to 1). Dotted lines represent the CES responses with $\phi = 1.5$, while each solid line represents responses for different values of the Kimball 'curvature' parameter ϵ, with $\sigma = 1.5$. All consumers have symmetric preferences with home bias.

R goods, financing it by selling Home goods. This triggers an increase in the imports of F and R goods when ϵ is high enough—$\epsilon = 5$ in Figure 2. Since both foreign countries, F and R, are completely symmetric in this example, the responses of both imports $c_F^{(H)}$ and $c_R^{(H)}$ are identical.

4.1.3 Nested CES with $N > 2$ Countries

If $N > 2$, then a nested-CES structure, with $(N-1)$ layers, does not generically give enough flexibility to replicate the dynamics of alternative aggregators.

So far we have compared Kimball to a single-layer Armington aggregator, which implied by definition that the bilateral elasticities were the same across all country-good pairs. One way to gain flexibility, while retaining the tractability of the Armington aggregator, is to move to a nested-CES framework. This will imply $N-1$ layers, which allows for $N-1$ elasticity parameters instead of a single one. The question becomes whether this framework can replicate any alternative to the Armington aggregator, by matching all of the bilateral elasticities and expenditure shares.

The answer to this question is generically no, the nested-CES structure does not allow enough degrees of freedom to fully match the first-order dynamics with alternative aggregators when we have more than two countries. As with a single-layer CES, we can easily adjust the consumption share parameters to match the steady-state expenditure shares of each good in each country. However, we have N bilateral elasticities to match for each country, but only $N-1$ nested-CES elasticity parameters, and are therefore missing one degree of freedom. With nested-CES, the bilateral elasticities between each country-good pair are combinations of the parameters in each CES layer. This means that we can set the elasticity parameter in each layer recursively to match all steady-state bilateral elasticities but one. For any given aggregator, there is a relationship between the different elasticities, which implies that, knowing $(N-1)$ elasticities, we can recover the remaining N^{th} elasticity. However, this relationship is specific to the aggregator. Hence, having matched $(N-1)$ steady-state bilateral elasticities from the alternative aggregator, using the $(N-1)$ elasticity parameters in the nested CES, does not ensure that the remaining N^{th} steady-state bilateral elasticity will be the same with the nested CES as with the alternative aggregator. This means that we cannot match all of the sufficient statistics given by Corollary 2, and so we cannot match the first-order dynamics.

Numerical Exercise with $N = 3$. As a concrete example, we return to the three-country setup presented in Section 4.1.2. We consider in each country $n \in \{H, F, R\}$, a nested-CES specification where the aggregate consumption is a CES aggregate of the locally produced good, and a bundle of the two imported goods. This implies the following specification and characteristics for country H:

$$C_t^{(H)} = f\left(c_{H,t}^{(H)}, c_{F,t}^{(H)}, c_{R,t}^{(H)}\right) = \left(a_H^{(H)\frac{1}{\phi_H}} c_{H,t}^{(H)\frac{\phi_H-1}{\phi_H}} + \left(1-a_H^{(H)}\right)^{\frac{1}{\phi_H}} C_{FR,t}^{(H)\frac{\phi_H-1}{\phi_H}}\right)^{\frac{\phi_H}{\phi_H-1}}$$

$$\text{where} \qquad C_{FR,t}^{(H)} = \left(a_F^{(H)\frac{1}{\phi_{FR}}} c_{F,t}^{(H)\frac{\phi_{FR}-1}{\phi_{FR}}} + \left(1-a_F^{(H)}\right)^{\frac{1}{\phi_{FR}}} C_{R,t}^{(H)\frac{\phi_{FR}-1}{\phi_{FR}}}\right)^{\frac{\phi_{FR}}{\phi_{FR}-1}}$$

The steady-state bilateral elasticities in country H become:[13]

$$\overline{\Phi}_{H,F}^{(H)} = \frac{\phi_H \phi_{FR}\left(\overline{\alpha}_F^{(H)} + \overline{\alpha}_H^{(H)}\overline{\alpha}_R^{(H)}\right)}{\overline{\alpha}_F^{(H)}\phi_{FR} + \overline{\alpha}_H^{(H)}\overline{\alpha}_R^{(H)}\phi_H}$$

$$\overline{\Phi}_{H,R}^{(H)} = \frac{\phi_H \phi_{FR}\left(\overline{\alpha}_R^{(H)} + \overline{\alpha}_H^{(H)}\overline{\alpha}_F^{(H)}\right)}{\overline{\alpha}_R^{(H)}\phi_{FR} + \overline{\alpha}_H^{(H)}\overline{\alpha}_F^{(H)}\phi_H}$$

$$\overline{\Phi}_{F,R}^{(H)} = \phi_{FR}$$

Suppose we want to set the parameters of these two CES aggregators so as to match the steady-state consumption shares and bilateral trade elasticities from a given parameterisation of the Kimball aggregator, with asymmetries, in order to replicate the first-order dynamics of the model. We can set the share parameters, $a_H^{(H)}$ and $a_F^{(H)}$, to match the steady-state consumption shares obtained from the Kimball aggregator directly. However, we now have two CES elasticity parameters, ϕ_H and ϕ_{FR}, to match the three bilateral elasticities.

In the specific case considered here, with symmetry across countries F and R, their steady-state

[13] See Appendix D for full derivations.

consumption shares in country H are equal, $\overline{\alpha}_F^{(H)} = \overline{\alpha}_R^{(H)}$, which implies that their bilateral elasticities are also equal, $\overline{\Phi}_{H,R}^{(H)} = \overline{\Phi}_{H,F}^{(H)}$. As this is true in both the Kimball and the nested-CES specifications, this allows us to match the country H first-order relative demand equations using a nested-CES aggregator. However, this is not true any more when turning to country F. The endowment asymmetry across our three countries implies asymmetric steady-state consumption shares and bilateral elasticities in country F, as stated in Table 2. In other words, $\overline{\alpha}_H^{(F)} \neq \overline{\alpha}_R^{(F)}$. After matching country F's steady-state consumption shares, and two of its bilateral elasticities, we have no degree of freedom left to ensure that the third Kimball bilateral elasticity is matched by the nested-CES specification, and the nested-CES steady-state bilateral elasticity $\overline{\Phi}_{H,F}^{(F)}$ is not equal to the Kimball one. Consequently, a nested-CES specification is not flexible enough to match the first-order dynamics of our Kimball setup, due to the endowment asymmetry.

Table 2: Steady-State Expenditure Shares and Elasticities of Substitution: Nested CES vs. Kimball Aggregator

	Nested CES	Kimball ($\epsilon = 5$)
Expenditure Shares		
$\overline{\alpha}_H^{(F)}$	9.91	9.91
$\overline{\alpha}_F^{(F)}$	74.19	74.19
$\overline{\alpha}_R^{(F)}$	15.90	15.90
Matched Elasticities		
$\overline{\Phi}_{H,R}^{(F)}$	2.52	2.52
$\overline{\Phi}_{F,R}^{(F)}$	1.11	1.11
Derived Elasticity		
$\overline{\Phi}_{H,F}^{(F)}$	1.36	5.59

4.2 Non-Homothetic Preferences

Theorem 1 and its corollaries also have implications for non-homothetic preferences, meaning if the aggregator is not HOD(1).

Notice that saying the aggregator is not HOD(1) can mean two things: that it is HOD(h), for $h \neq 1$, or that it is non-homogeneous. We focus on the former case. While Theorem 1 involves all ratios $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$ for all $i = 1, 2, ...N$, recall that Corollary 1, by focusing specifically on homogeneous functions, only depends on h. It is useful to again consider the two cases.

4.2.1 Case 1: Two Countries

If $N = 2$, then any HOD(h) aggregator, $h \in \mathbb{R}$, is equivalent at first order to a generalised Armington-style aggregator that is HOD(h), with the same steady-state elasticity and consumption shares.

We can define a HOD(h) generalisation of the 2-good Armington aggregator:

$$C_t^{(n)} \equiv f(c_{1,t}^{(n)}, c_{2,t}^{(n)}) = \left(a_1^{(n)\frac{1}{\phi}} c_{1,t}^{(n)\frac{\phi-1}{\phi}} + \left(1 - a_1^{(n)}\right)^{\frac{1}{\phi}} c_{2,t}^{(n)\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}h}$$

The parameters ϕ and $a_1^{(n)}$ have the same interpretation as before, and h is a free parameter that determines the degree of homogeneity. Note that, with this generalised aggregator, the relative demand function remains the same as in the standard case.

This means that any model that uses an alternative aggregator can be mapped parsimoniously into this generalised Armington aggregator by setting the parameters ϕ and $a_1^{(n)}$ to match the steady-state elasticity of substitution and consumption shares, as before, and setting h equal to the same degree of homogeneity as the alternative aggregator.

Notice again that, while these results show that deviating from HOD(1) aggregators can affect the first-order dynamics, even with $N = 2$, they also specify that the first-order effect of any HOD(h) aggregator relative to the standard Armington model is determined entirely by a single parameter, h.

4.2.2 Case 2: $N > 2$ Countries

If $N > 2$, then alternative HOD(h) aggregators can create differences with respect to the N-good generalised HOD(h) Armington aggregator, by allowing bilateral elasticities of substitution to be different across different pairs of goods in steady state.

The N-good generalised HOD(h) Armington aggregator can be defined as:

$$C_t^{(n)} = f(c_{1,t}^{(n)}, \dots c_{N,t}^{(n)}) = \left(\sum_{i=1}^{N} a_i^{(n)\frac{1}{\phi}} c_{i,t}^{(n)\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}h}$$

where $\sum_{i=1}^{N} a_i^{(n)} = 1$.

The same reasoning as the HOD(1) case can be applied here, again with the addition that the parameter h is chosen correctly. Note, once again, that the additional mechanism that alternative HOD(h) aggregators bring when $N > 2$ is only through the cross-elasticity differences in steady state. These differences are also the ones that prevent matching a HOD(h) aggregator with a generalised nested-CES specification, using the generalised Armington.

5 Conclusions

We have shown that the first-order dynamics of models that aggregate goods from multiple countries into one consumption bundle can be summarised by sufficient statistics that reflect the characteristics of the aggregation function. These sufficient statistics include the steady-state values of consumption shares, bilateral elasticities and ratios related to the degree of homogeneity of the aggregator. This main result can be unpacked into a number of more specific implications.

First, in a two-country model, the standard Armington aggregator is equivalent at first order to any other aggregator that is homogeneous of degree one, with the same elasticity of substitution and consumption expenditure shares in steady state. We have also put forward a parsimonious generalisation of the Armington aggregator that is homogeneous of arbitrary degree, h. In a two-country setup again, this generalised Armington aggregator is equivalent at first order to any aggregator with the same elasticity of substitution and consumption expenditure shares in steady state, and the same degree of homogeneity.

Second, when the number of countries, N, is larger than two, the Armington aggregator can become restrictive to the extent that it imposes that the bilateral elasticities of substitution of each pair of goods are given by the same parameter. Other aggregators that allow these elasticities to be different in steady state can therefore affect the first-order dynamics of the model. However, again, this implies that the channel through which these aggregators affect the model is captured entirely by the asymmetries in the steady-state pair-wise elasticities of substitution. We also showed that a nested-CES structure, nesting $(N-1)$ Armington aggregators, does not provide enough degrees of freedom to generically replicate alternative aggregators. Similarly, when compared to an alternative aggregator that is homogeneous of degree h, our generalised Armington aggregator can replicate the first-order dynamics under symmetry, but not under asymmetry, due to the differences in the steady-state elasticities of substitution across different pairs of goods.

Notice that throughout the results, only the steady-state elasticities of substitution affected the first-order dynamics of the model. This means that one of the standard mechanisms that many alternative aggregators are used to capture in dynamic models—varying elasticities of substitution across time—does not have a first-order effect in these models.

For clarity, the model we laid out at in Section 2 was a simple endowment economy. However, Theorem 1, and its corollaries, would continue to hold if we introduced a perfectly competitive production sector, which does not take the demand structure into account. This is because the intratemporal consumption-demand block of the model, which is the block which depends on the aggregator function, remains separate to the production side of the model. Moreover, in that case, while we have focused here on the *consumption* aggregator, the same results would hold if we looked at the aggregators used for other types of goods, such as intermediate inputs or investment goods, so long as the optimal composition of these goods, between domestic and foreign goods, remains an intratemporal problem.

Finally, we derived all of these results analytically in a linearised model, and so they hold exactly at first order. This means that, in principle, the alternative aggregators may have further effects on the dynamics of the model at higher orders. However, the standard workhorse NOEM model is very close to being linear, meaning that these higher-order effects are small by definition, especially for the standard size of shocks. We leave it for future research to explore the impact of the trade aggregator choice in different settings in which non-linearities and higher-order effects may matter more.

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Appendix

A Proof of Theorem 1

For each country n, the relevant system of equations that are affected by the aggregator function, f, and the individual consumption levels, $\mathbf{c}_t^{(n)}$, are the definition of aggregate consumption, the definition of the price index, the $N-1$ relative demand functions, and the goods market clearing condition:

$$\begin{aligned} C^{(n)} &= f(\mathbf{c}^{(n)}) \\ C^{(n)} &= \sum_{i=1}^{N} p_i^{(n)} c_i^{(n)} \\ \frac{p_i^{(n)}}{p_N^{(n)}} &= \frac{f_i^{(n)}}{f_N^{(n)}} \quad \forall\, i = 1, \dots N-1 \\ Y^{(n)} &= \sum_{i=1}^{N} c_n^{(i)} \end{aligned}$$

where we have dropped the time subscripts for simplicity. In what follows, we will also drop the country (n) superscripts for simplicity, except for the derivations related to the goods market clearing condition where they are relevant.

We want to derive the log-linear form of these equations to understand what drives the first-order dynamics, and in particular how it depends on the function f. To do this, we will apply the general formula for the first-order Taylor expansion. Write each equation in a generic format $F(\mathbf{x}) = 0$, where $\mathbf{x}$ is the vector of all model variables. Then the multivariate first-order Taylor expansion around a point $\overline{\mathbf{x}}$ is given by:

$$\begin{aligned} F(\mathbf{x}) &\approx \left(F'(\mathbf{x})\,|_{\mathbf{x}=\overline{\mathbf{x}}}\right)' (\mathbf{x} - \overline{\mathbf{x}}) \\ &= \sum_i \left.\frac{\partial F(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} (x_i - \overline{x}_i) \\ &= \sum_i F_i(\overline{\mathbf{x}}) \overline{x}_i \widetilde{x}_i \end{aligned}$$

where we use the notation $\widetilde{x} \equiv (x - \overline{x})/\overline{x}$, where $\overline{x}$ denotes the steady state. We will apply this formula to each of the equations above.

Aggregate Consumption

$$\begin{aligned}
C &= f(\mathbf{c}) \\
0 &= C - f(\mathbf{c}) \\
&\approx \overline{C}\widetilde{C} - \sum_{i=1}^{N} \overline{f}_i \overline{c}_i \widetilde{c}_i \\
&= \widetilde{C} - \sum_{i=1}^{N} \frac{\overline{f}_i \overline{c}_i}{\overline{C}} \widetilde{c}_i \\
\widetilde{C} &\approx \sum_{i=1}^{N} \frac{\overline{f}_i \overline{c}_i}{\overline{C}} \widetilde{c}_i
\end{aligned}$$

To simplify this equation, recall the FOCs of the cost-minimisation problem defined above:

$$Pp_i = \lambda f_i \quad \forall\, i = 1, \ldots N$$

We can solve for the Lagrange multiplier using the definition of the aggregate price index:

$$\begin{aligned}
C &= \sum_{i=1}^{N} p_i c_i \\
&= \sum_{i=1}^{N} \frac{\lambda f_i}{P} c_i \\
&= \frac{\lambda}{P} \sum_{i=1}^{N} f_i c_i \\
\lambda &= \frac{P}{\mathcal{H}(\mathbf{c})}
\end{aligned}$$

where

$$\mathcal{H}(\mathbf{c}) = \frac{\sum_{i=1}^{N} f_i c_i}{C} = \frac{\sum_{i=1}^{N} f_i c_i}{f(\mathbf{c})}$$

Plugging this into the FOCs:

$$Pp_i = \frac{P}{\mathcal{H}(\mathbf{c})} f_i$$

or

$$p_i = \frac{f_i}{\mathcal{H}(\mathbf{c})} \qquad \Rightarrow \qquad f_i = p_i \mathcal{H}(\mathbf{c})$$

Hence

$$\frac{\overline{f}_i \overline{c}_i}{\overline{C}} = \mathcal{H}(\overline{\mathbf{c}}) \frac{\overline{p}_i \overline{c}_i}{\overline{C}}$$

Define the steady state share of consumption expenditure on good j:

$$\alpha_i \equiv \frac{\overline{p}_i \overline{c}_i}{\overline{C}}$$

Putting these together, denoting $\mathcal{H}(\overline{\mathbf{c}}) \equiv \overline{\mathcal{H}}$, the linearised form of the aggregator is given by:

$$\widetilde{C} \approx \overline{\mathcal{H}} \sum_{i=1}^{N} \alpha_i \widetilde{c}_i$$

This depends on $\overline{\mathcal{H}}$ and α_i for $i = 1, \ldots N$.

Consumer Price Index

$$\begin{aligned}
C &= \sum_{i=1}^{N} p_i c_i \\
0 &= C - \sum_{i=1}^{N} p_i c_i \\
&\approx \overline{C}\widetilde{C} - \sum_{i=1}^{N} \overline{p}_i \overline{c}_i \widetilde{c}_i - \sum_{i=1}^{N} \overline{c}_i \overline{p}_i \widetilde{p}_i \\
&\approx \widetilde{C} - \sum_{i=1}^{N} \frac{\overline{p}_i \overline{c}_i}{\overline{C}} \widetilde{c}_i - \sum_{i=1}^{N} \frac{\overline{p}_i \overline{c}_i}{\overline{C}} \widetilde{p}_i \\
&\approx \widetilde{C} - \sum_{i=1}^{N} \alpha_i \widetilde{c}_i - \sum_{i=1}^{N} \alpha_i \widetilde{p}_i \\
&\approx \widetilde{C} - \frac{1}{\overline{\mathcal{H}}} \widetilde{C} - \sum_{i=1}^{N} \alpha_i \widetilde{p}_i \\
\frac{\overline{\mathcal{H}} - 1}{\overline{\mathcal{H}}} \widetilde{C} &\approx \sum_{i=1}^{N} \alpha_i \widetilde{p}_i
\end{aligned}$$

where the linearised form of the aggregator was used to simplify the equation. Again, this depends on $\overline{\mathcal{H}}$ and α_i for $i = 1, \ldots N$.

Relative Demand Functions

Consider a specific i without loss of generality:

$$\begin{aligned}
\frac{p_i}{p_N} &= \frac{f_i(\mathbf{c})}{f_N(\mathbf{c})} \\
0 &= \frac{p_i}{p_N} - \frac{f_i(\mathbf{c})}{f_N(\mathbf{c})} \\
0 &\approx \frac{1}{\overline{p}_N}\overline{p}_i\widetilde{p}_i - \frac{\overline{p}_i}{\overline{p}_N^2}\overline{p}_N\widetilde{p}_N - \sum_{k=1}^{N} \left.\frac{\partial\left(\frac{f_i}{f_N}\right)}{\partial c_k}\right|_{ss} \overline{c}_k\widetilde{c}_k \\
&= \frac{\overline{p}_i}{\overline{p}_N}\left(\widetilde{p}_i - \widetilde{p}_N\right) - \sum_{k=1}^{N} \left.\frac{\partial\left(\frac{f_i}{f_N}\right)}{\partial c_k}\right|_{ss} \overline{c}_k\widetilde{c}_k
\end{aligned}$$

Consider the partial derivative term:

$$\begin{aligned}
\frac{\partial\left(\frac{f_i}{f_N}\right)}{\partial c_k} &= \frac{1}{f_N}\frac{\partial f_i}{\partial c_k} - \frac{f_i}{f_N^2}\frac{\partial f_N}{\partial c_k} \\
&= \frac{f_{ik}}{f_N} - \frac{f_i f_{Nk}}{f_N^2} \\
&= \frac{f_i}{f_N}\left(\frac{f_{ik}}{f_i} - \frac{f_{Nk}}{f_N}\right)
\end{aligned}$$

Plugging this back in:

$$\begin{aligned}
0 &\approx \frac{\overline{p}_i}{\overline{p}_N}\left(\widetilde{p}_i - \widetilde{p}_N\right) - \sum_{k=1}^{N} \frac{f_i}{f_N}\left.\left(\frac{f_{ik}}{f_i} - \frac{f_{Nk}}{f_N}\right)\right|_{ss} \overline{c}_k\widetilde{c}_k \\
&= \frac{\overline{p}_i}{\overline{p}_N}\left(\widetilde{p}_i - \widetilde{p}_N\right) - \frac{\overline{f}_i}{\overline{f}_N}\sum_{k=1}^{N}\left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k\widetilde{c}_k
\end{aligned}$$

Using the fact that $\overline{p}_i/\overline{p}_N = \overline{f}_i/\overline{f}_N$:

$$\widetilde{p}_i - \widetilde{p}_N \approx \sum_{k=1}^{N}\left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k\widetilde{c}_k = \sum_{k=1}^{N} coef_k^{(iN)}\widetilde{c}_k$$

where $coef_k^{(iN)} \equiv \left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k$.

Consider now the definition of the elasticity of substitution between two different goods x and y (we consider here the direct partial elasticity as defined by McFadden (1963) or Sato (1967)):

$$\Phi_{xy} = \frac{\partial \ln\left(c_x/c_y\right)}{\partial \ln\left(f_y/f_x\right)} = -\left(\frac{1}{c_x f_x} + \frac{1}{c_y f_y}\right)\left[\left(\frac{f_{xx}}{f_x^2} - \frac{f_{xy}}{f_x f_y}\right) + \left(\frac{f_{yy}}{f_y^2} - \frac{f_{xy}}{f_x f_y}\right)\right]^{-1}$$

In a first step, we derive some relationships between the coefficients of the linearised relative demand function, the steady state bilateral elasticities and the steady state consumption shares.

$$
\begin{aligned}
\overline{\Phi}_{iN}^{-1} &= -\left[\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2} - \frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}\right) + \left(\frac{\overline{f}_{NN}}{\overline{f}_N^2} - \frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}\right)\right]\left(\frac{1}{\overline{c}_i\overline{f}_i} + \frac{1}{\overline{c}_N\overline{f}_N}\right)^{-1} \\
&= -\left[\frac{1}{\overline{f}_i\overline{c}_i}coef_i^{(iN)} - \frac{1}{\overline{f}_N\overline{c}_N}coef_N^{(iN)}\right]\left(\frac{1}{\overline{c}_i\overline{f}_i} + \frac{1}{\overline{c}_N\overline{f}_N}\right)^{-1} \\
&= -\left[\frac{1}{\overline{f}_i\overline{c}_i}coef_i^{(iN)} - \frac{1}{\overline{f}_N\overline{c}_N}coef_N^{(iN)}\right]\frac{\overline{c}_i\overline{c}_N\overline{f}_i\overline{f}_N}{\overline{c}_N\overline{f}_N + \overline{c}_i\overline{f}_i} \\
&= -\frac{\overline{c}_N\overline{f}_N}{\overline{c}_i\overline{f}_i + \overline{c}_N\overline{f}_N}coef_i^{(iN)} + \frac{\overline{c}_i\overline{f}_i}{\overline{c}_i\overline{f}_i + \overline{f}_N\overline{c}_N}coef_N^{(iN)}
\end{aligned}
$$

Using the definition of the steady state expenditure shares:

$$
\begin{aligned}
\frac{\overline{c}_i\overline{f}_i}{\overline{c}_i\overline{f}_i + \overline{f}_N\overline{c}_N} &= \frac{\overline{c}_i\frac{\overline{f}_i}{\overline{f}_N}}{\overline{c}_i\frac{\overline{f}_i}{\overline{f}_N} + \overline{c}_N} = \frac{\overline{c}_i\frac{\overline{p}_i}{\overline{p}_N}}{c_i\frac{\overline{p}_i}{\overline{p}_N} + \overline{c}_N} \\
&= \frac{\overline{c}_i\overline{p}_i}{\overline{c}_i\overline{p}_i + \overline{p}_N\overline{c}_N} = \frac{\overline{c}_i\overline{p}_i}{\sum_l \overline{c}_l\overline{p}_l}\frac{\sum_l \overline{c}_l\overline{p}_l}{\overline{c}_i\overline{p}_i + \overline{p}_N\overline{c}_N} \\
&= \frac{\alpha_i}{\alpha_i + \alpha_N}
\end{aligned}
$$

And we obtain:

$$
\overline{\Phi}_{iN}^{-1} = -\frac{\alpha_N}{\alpha_i + \alpha_N}coef_i^{(iN)} + \frac{\alpha_i}{\alpha_i + \alpha_N}coef_N^{(iN)}
$$

i.e.

$$
(\alpha_i + \alpha_N)\overline{\Phi}_{iN}^{-1} = -\alpha_N coef_i^{(iN)} + \alpha_i coef_N^{(iN)} \tag{12}
$$

Equation (12) is the first type of relationship we were aiming for, and is true for every $i = 1, 2, ..., N-1$. Now we derive a second type of relationship, involving two bilateral elasticities.

$$
\begin{aligned}
&\left(\frac{1}{\overline{c}_i\overline{f}_i}+\frac{1}{\overline{c}_k\overline{f}_k}\right)\Phi_{ik}^{-1}-\left(\frac{1}{\overline{c}_N\overline{f}_N}+\frac{1}{\overline{c}_k\overline{f}_k}\right)\overline{\Phi}_{Nk}^{-1}\\
&\qquad=-\left[\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2}-\frac{\overline{f}_{ik}}{\overline{f}_i\overline{f}_k}\right)+\left(\frac{\overline{f}_{kk}}{\overline{f}_k^2}-\frac{\overline{f}_{ik}}{\overline{f}_i\overline{f}_k}\right)\right]\\
&\qquad\quad+\left[\left(\frac{\overline{f}_{NN}}{\overline{f}_N^2}-\frac{\overline{f}_{Nk}}{\overline{f}_N\overline{f}_k}\right)+\left(\frac{\overline{f}_{kk}}{\overline{f}_k^2}-\frac{\overline{f}_{Nk}}{\overline{f}_N\overline{f}_k}\right)\right]\\
&\qquad=-\frac{1}{\overline{f}_i}\left(\frac{\overline{f}_{ii}}{\overline{f}_i}-\frac{\overline{f}_{ik}}{\overline{f}_k}\right)-\frac{1}{\overline{f}_k}\left(\frac{\overline{f}_{kk}}{\overline{f}_k}-\frac{\overline{f}_{ik}}{\overline{f}_i}\right)\\
&\qquad\quad+\frac{1}{\overline{f}_N}\left(\frac{\overline{f}_{NN}}{\overline{f}_N}-\frac{\overline{f}_{Nk}}{\overline{f}_k}\right)+\frac{1}{\overline{f}_k}\left(\frac{\overline{f}_{kk}}{\overline{f}_k}-\frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\\
&\qquad=-\frac{1}{\overline{f}_i}\left(\frac{\overline{f}_{ii}}{\overline{f}_i}-\frac{\overline{f}_{ik}}{\overline{f}_k}\right)+\frac{1}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}+\frac{1}{\overline{f}_N}\left(\frac{\overline{f}_{NN}}{\overline{f}_N}-\frac{\overline{f}_{Nk}}{\overline{f}_k}\right)\\
&\qquad=-\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2}-\frac{\overline{f}_{ik}}{\overline{f}_i\overline{f}_k}\right)+\frac{1}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}+\left(\frac{\overline{f}_{NN}}{\overline{f}_N^2}-\frac{\overline{f}_{Nk}}{\overline{f}_N\overline{f}_k}\right)\\
&\qquad=-\frac{\overline{f}_{ii}}{\overline{f}_i^2}+\frac{1}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}+\frac{\overline{f}_{NN}}{\overline{f}_N^2}+\frac{1}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}\\
&\qquad=-\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2}-\frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}\right)-\left(\frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}-\frac{\overline{f}_{NN}}{\overline{f}_N^2}\right)+\frac{2}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}\\
&\qquad=-\frac{1}{\overline{c}_i\overline{f}_i}coef_i^{(iN)}-\frac{1}{\overline{c}_N\overline{f}_N}coef_N^{(iN)}+\frac{2}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}
\end{aligned}
$$

Now bringing back expenditure shares as above:

$$
\begin{aligned}
&\left(\frac{\sum_l\overline{c}_l\overline{f}_l}{c_i\overline{f}_i}+\frac{\sum_l\overline{c}_l\overline{f}_l}{\overline{c}_k\overline{f}_k}\right)\overline{\Phi}_{ik}^{-1}-\left(\frac{\sum_l\overline{c}_l\overline{f}_l}{\overline{c}_N\overline{f}_N}+\frac{\sum_l\overline{c}_l\overline{f}_l}{\overline{c}_k\overline{f}_k}\right)\overline{\Phi}_{Nk}^{-1}\\
&\qquad=-\frac{\sum_l\overline{c}_l\overline{f}_l}{\overline{c}_i\overline{f}_i}coef_i^{(iN)}-\frac{\sum_l\overline{c}_l\overline{f}_l}{\overline{c}_N\overline{f}_N}coef_N^{(iN)}+\frac{2\sum_l\overline{c}_l\overline{f}_l}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{1}{\alpha_i}+\frac{1}{\alpha_k}\right)\overline{\Phi}_{ik}^{-1}-\left(\frac{1}{\alpha_N}+\frac{1}{\alpha_k}\right)\overline{\Phi}_{Nk}^{-1}\\
&\qquad=-\frac{1}{\alpha_i}coef_i^{(iN)}-\frac{1}{\alpha_N}coef_N^{(iN)}+\frac{2}{\alpha_k}coef_k^{(iN)}\\
&(\alpha_i+\alpha_k)\alpha_N\overline{\Phi}_{ik}^{-1}-(\alpha_k+\alpha_N)\alpha_i\overline{\Phi}_{Nk}^{-1}\\
&\qquad=-\alpha_k\alpha_N coef_i^{(iN)}-\alpha_i\alpha_k coef_N^{(iN)}+2\alpha_i\alpha_N coef_k^{(iN)} \qquad (13)
\end{aligned}
$$

Equation (13) is our second type of relationship, and is valid for all $i=1,2,...,N-1$ and for all $k\neq i,N$. Now, we can use the relationships obtained in equations (12) and (13) to express the linearised relative demand function as a function of the steady state expenditure shares, elasticities and ratios $\mathcal{H}$ and $\mathcal{H}_j$.

From equation (12), we have:

$$coef_N^{(iN)} = \frac{\alpha_i + \alpha_N}{\alpha_i}\overline{\Phi}_{iN}^{-1} + \frac{\alpha_N}{\alpha_i} coef_i^{(iN)} \tag{14}$$

And from equation (13), for all $k \neq i, N$:

$$\begin{aligned}
coef_k^{(iN)} =& \frac{\alpha_i + \alpha_k}{2\alpha_i}\overline{\Phi}_{ik}^{-1} - \frac{\alpha_k + \alpha_N}{2\alpha_N}\overline{\Phi}_{Nk}^{-1} + \frac{\alpha_k}{2\alpha_i} coef_i^{(iN)} + \frac{\alpha_k}{2\alpha_N} coef_N^{(iN)} \\
=& \frac{\alpha_i + \alpha_k}{2\alpha_i}\overline{\Phi}_{ik}^{-1} - \frac{\alpha_k + \alpha_N}{2\alpha_N}\overline{\Phi}_{Nk}^{-1} + \frac{\alpha_k}{2\alpha_i} coef_i^{(iN)} \\
&+ \frac{\alpha_k}{2\alpha_N}\left(\frac{\alpha_i + \alpha_N}{\alpha_i}\overline{\Phi}_{iN}^{-1} + \frac{\alpha_N}{\alpha_i} coef_i^{(iN)}\right) \\
=& \frac{\alpha_k}{\alpha_i} coef_i^{(iN)} + \frac{\alpha_i + \alpha_k}{2\alpha_i}\overline{\Phi}_{ik}^{-1} - \frac{\alpha_k + \alpha_N}{2\alpha_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i + \alpha_N)\alpha_k}{2\alpha_N\alpha_i}\overline{\Phi}_{iN}^{-1}
\end{aligned} \tag{15}$$

Plugging expressions (14) and (15) in the linearised relative demand function:

$$\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N =& coef_i^{(iN)}\widetilde{c}_i + \sum_{k=1, k\neq i}^{N-1} coef_k^{(iN)}\widetilde{c}_k + coef_N^{(iN)}\widetilde{c}_N \\
=& coef_i^{(iN)}\widetilde{c}_i \\
&+ \sum_{k=1, k\neq i}^{N-1}\left(\frac{\alpha_k}{\alpha_i} coef_i^{(iN)} + \frac{\alpha_i + \alpha_k}{2\alpha_i}\overline{\Phi}_{ik}^{-1} - \frac{\alpha_k + \alpha_N}{2\alpha_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i + \alpha_N)\alpha_k}{2\alpha_N\alpha_i}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_k \\
&+ \left(\frac{\alpha_i + \alpha_N}{\alpha_i}\overline{\Phi}_{iN}^{-1} + \frac{\alpha_N}{\alpha_i} coef_i^{(iN)}\right)\widetilde{c}_N \\
=& coef_i^{(iN)}\widetilde{c}_i + \frac{\alpha_N}{\alpha_i} coef_i^{(iN)}\widetilde{c}_N + \sum_{k=1, k\neq i}^{N-1}\left(\frac{\alpha_k}{\alpha_i} coef_i^{(iN)}\right)\widetilde{c}_k \\
&+ \sum_{k=1, k\neq i}^{N-1}\left(\frac{\alpha_i + \alpha_k}{2\alpha_i}\overline{\Phi}_{ik}^{-1} - \frac{\alpha_k + \alpha_N}{2\alpha_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i + \alpha_N)\alpha_k}{2\alpha_N\alpha_i}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_k \\
&+ \frac{\alpha_i + \alpha_N}{\alpha_i}\overline{\Phi}_{iN}^{-1}\widetilde{c}_N \\
=& \frac{1}{\alpha_i} coef_i^{(iN)}\left(\alpha_i\widetilde{c}_i + \alpha_N\widetilde{c}_N + \sum_{k=1, k\neq i}^{N-1}\alpha_k\widetilde{c}_k\right) \\
&+ \sum_{k=1, k\neq i}^{N-1}\left(\frac{\alpha_i + \alpha_k}{2\alpha_i}\overline{\Phi}_{ik}^{-1} - \frac{\alpha_k + \alpha_N}{2\alpha_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i + \alpha_N)\alpha_k}{2\alpha_N\alpha_i}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_k \\
&+ \frac{\alpha_i + \alpha_N}{\alpha_i}\overline{\Phi}_{iN}^{-1}\widetilde{c}_N \\
=& \frac{1}{\alpha_i}\Bigg[coef_i^{(iN)}\left(\sum_{k=1}^{N}\alpha_k\widetilde{c}_k\right) \\
&+ \frac{1}{2}\sum_{k=1, k\neq i}^{N-1}\left((\alpha_i + \alpha_k)\overline{\Phi}_{ik}^{-1} - \frac{(\alpha_k + \alpha_N)\alpha_i}{\alpha_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i + \alpha_N)\alpha_k}{\alpha_N}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_k \\
&+ (\alpha_i + \alpha_N)\overline{\Phi}_{iN}^{-1}\widetilde{c}_N\Bigg]
\end{aligned}$$

From the aggregate consumption linearisation we know that:

$$\widetilde{C} \approx \overline{\mathcal{H}} \sum_{k=1}^{N} \alpha_k \widetilde{c}_k$$

So we get:

$$\begin{aligned}
\alpha_i \left(\widetilde{p}_i - \widetilde{p}_N\right) = coef_i^{(iN)} \frac{\widetilde{C}}{\overline{\overline{\mathcal{H}}}} \\
+ \frac{1}{2} \sum_{k=1, k \neq i}^{N-1} \left(\left(\alpha_i + \alpha_k\right) \overline{\Phi}_{ik}^{-1} - \frac{(\alpha_k + \alpha_N)\alpha_i}{\alpha_N} \overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i + \alpha_N)\alpha_k}{\alpha_N} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_k \\
+ (\alpha_i + \alpha_N) \overline{\Phi}_{iN}^{-1} \widetilde{c}_N
\end{aligned} \tag{16}$$

With a similar approach, still using equations (12) and (13), we can obtain the following expressions for the coefficients and the linearised relative demand function:

$$\begin{aligned}
coef_i^{(iN)} &= \frac{\alpha_i}{\alpha_N} coef_N^{(iN)} - \frac{\alpha_i + \alpha_N}{\alpha_N} \overline{\Phi}_{iN}^{-1} \\
coef_k^{(iN)} &= \frac{\alpha_k}{\alpha_N} coef_N^{(iN)} + \frac{\alpha_i + \alpha_k}{2\alpha_i} \overline{\Phi}_{ik}^{-1} - \frac{\alpha_k + \alpha_N}{2\alpha_N} \overline{\Phi}_{Nk}^{-1} - \frac{\alpha_k(\alpha_i + \alpha_N)}{2\alpha_i \alpha_N} \overline{\Phi}_{iN}^{-1} \quad \forall k \neq i, N
\end{aligned}$$

implying:

$$\begin{aligned}
\alpha_N \left(\widetilde{p}_i - \widetilde{p}_N\right) = coef_N^{(iN)} \frac{\widetilde{C}}{\overline{\overline{\mathcal{H}}}} \\
+ \frac{1}{2} \sum_{k=1, k \neq i}^{N-1} \left(\frac{(\alpha_i + \alpha_k)\alpha_N}{\alpha_i} \overline{\Phi}_{ik}^{-1} - \left(\alpha_k + \alpha_N\right) \overline{\Phi}_{Nk}^{-1} - \frac{\alpha_k(\alpha_i + \alpha_N)}{\alpha_i} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_k \\
- (\alpha_i + \alpha_N) \overline{\Phi}_{iN}^{-1} \widetilde{c}_i
\end{aligned} \tag{17}$$

Using again a similar approach, from equations (12) and (13):

$$\begin{aligned}
coef_N^{(iN)} =& \frac{\alpha_k + \alpha_N}{2\alpha_k} \overline{\Phi}_{Nk}^{-1} - \frac{(\alpha_i + \alpha_k)\alpha_N}{2\alpha_i \alpha_k} \overline{\Phi}_{ik}^{-1} + \frac{\alpha_i + \alpha_N}{2\alpha_i} \overline{\Phi}_{iN}^{-1} \\
&+ \frac{\alpha_N}{\alpha_k} coef_k^{(iN)} \quad \forall k \neq i, N
\end{aligned} \tag{18}$$

$$\begin{aligned}
coef_i^{(iN)} =& \frac{\alpha_i}{\alpha_N} coef_N^{(iN)} - \frac{\alpha_i + \alpha_N}{\alpha_N} \overline{\Phi}_{iN}^{-1} \\
=& \frac{\alpha_i}{\alpha_k} coef_k^{(iN)} + \frac{\alpha_i(\alpha_k + \alpha_N)}{2\alpha_k \alpha_N} \overline{\Phi}_{Nk}^{-1} - \frac{\alpha_i + \alpha_k}{2\alpha_k} \overline{\Phi}_{ik}^{-1} \\
&- \frac{\alpha_i + \alpha_N}{2\alpha_N} \overline{\Phi}_{iN}^{-1} \quad \forall k \neq i, N
\end{aligned} \tag{19}$$

Considering a specific $k \neq i, N$ without loss of generality, we now need to also express $coef_l^{(iN)}$

$(l \neq i, N, k)$ as a function of $coef_k^{(iN)}$, steady state bilateral elasticities and consumption shares. Rewriting equation (19) for any $l \neq i, N, k$:

$$coef_N^{(iN)} = \frac{\alpha_l + \alpha_N}{2\alpha_l}\overline{\Phi}_{Nl}^{-1} - \frac{(\alpha_i + \alpha_l)\alpha_N}{2\alpha_i\alpha_l}\overline{\Phi}_{il}^{-1} + \frac{\alpha_i + \alpha_N}{2\alpha_i}\overline{\Phi}_{iN}^{-1} + \frac{\alpha_N}{\alpha_l}coef_l^{(iN)}$$

Implying for all $l \neq i, N, k$:

$$\begin{aligned} coef_l^{(iN)} =& \frac{\alpha_l}{\alpha_N}\left(coef_N^{(iN)} + \frac{(\alpha_i + \alpha_l)\alpha_N}{2\alpha_i\alpha_l}\overline{\Phi}_{il}^{-1} - \frac{\alpha_l + \alpha_N}{2\alpha_l}\overline{\Phi}_{Nl}^{-1} - \frac{\alpha_i + \alpha_N}{2\alpha_i}\overline{\Phi}_{iN}^{-1}\right) \\ =& \frac{\alpha_l}{\alpha_k}coef_k^{(iN)} + \frac{\alpha_l(\alpha_k + \alpha_N)}{2\alpha_k\alpha_N}\overline{\Phi}_{Nk}^{-1} \\ &- \frac{\alpha_l(\alpha_i + \alpha_k)}{2\alpha_i\alpha_k}\overline{\Phi}_{ik}^{-1} + \frac{(\alpha_i + \alpha_l)}{2\alpha_i}\overline{\Phi}_{il}^{-1} - \frac{\alpha_l + \alpha_N}{2\alpha_N}\overline{\Phi}_{Nl}^{-1} \end{aligned} \tag{20}$$

We can again plug the expressions (18) to (20) into the linearised relative demand function and obtain after some manipulations:

$$\begin{aligned} \alpha_k\left(\widetilde{p}_i - \widetilde{p}_N\right) =& coef_k^{(iN)}\frac{\widetilde{C}}{\overline{\overline{\mathcal{H}}}} \\ &+ \frac{1}{2}\left(\frac{\alpha_i(\alpha_k + \alpha_N)}{\alpha_N}\overline{\Phi}_{Nk}^{-1} - (\alpha_i + \alpha_k)\,\overline{\Phi}_{ik}^{-1} - \frac{(\alpha_i + \alpha_N)\alpha_k}{\alpha_N}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_i \\ &+ \frac{1}{2}\left((\alpha_k + \alpha_N)\,\overline{\Phi}_{Nk}^{-1} - \frac{(\alpha_i + \alpha_k)\alpha_N}{\alpha_i}\overline{\Phi}_{ik}^{-1} + \frac{(\alpha_i + \alpha_N)\alpha_k}{\alpha_i}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_N \\ &+ \frac{1}{2}\sum_{l=1,l\neq i,k}^{N-1}\left(\frac{\alpha_l(\alpha_k + \alpha_N)}{\alpha_N}\overline{\Phi}_{Nk}^{-1}\frac{\alpha_l(\alpha_i + \alpha_k)}{\alpha_i}\overline{\Phi}_{ik}^{-1}\right. \\ &+ \left.\frac{(\alpha_i + \alpha_l)\alpha_k}{\alpha_i}\overline{\Phi}_{il}^{-1} - \frac{(\alpha_l + \alpha_N)\alpha_k}{\alpha_N}\overline{\Phi}_{Nl}^{-1}\right)\widetilde{c}_l \end{aligned} \tag{21}$$

Equation (21) is valid for any $k \neq i, N$. Now let's sum equations (16), (17) and all (21) for all $k \neq i, N$ and notice that by definition of the consumption shares:

$$\alpha_i\left(\widetilde{p}_i - \widetilde{p}_N\right) + \alpha_N\left(\widetilde{p}_i - \widetilde{p}_N\right) + \sum_{k=1,k\neq i}^{N-1}\alpha_k\left(\widetilde{p}_i - \widetilde{p}_N\right) = \left(\widetilde{p}_i - \widetilde{p}_N\right)\sum_{k=1}^{N}\alpha_k = \widetilde{p}_i - \widetilde{p}_N$$

And we obtain the following expression for the linearised relative demand function:

$$
\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N = & \frac{\widetilde{C}}{\overline{\overline{\mathcal{H}}}} \sum_{k=1}^{N} \left(coef_k^{(iN)} \right) \\
& + \widetilde{c}_i \left[\frac{1}{2} \sum_{k=1, k \neq i}^{N-1} \left(\frac{\alpha_i (\alpha_k + \alpha_N)}{\alpha_N} \overline{\Phi}_{Nk}^{-1} - (\alpha_i + \alpha_k) \overline{\Phi}_{ik}^{-1} - \frac{(\alpha_i + \alpha_N)\alpha_k}{\alpha_N} \overline{\Phi}_{iN}^{-1} \right) \right. \\
& \left. - (\alpha_i + \alpha_N) \overline{\Phi}_{iN}^{-1} \right] \\
& + \widetilde{c}_N \left[\frac{1}{2} \sum_{k=1, k \neq i}^{N-1} \left((\alpha_k + \alpha_N) \overline{\Phi}_{Nk}^{-1} - \frac{(\alpha_i + \alpha_k)\alpha_N}{\alpha_i} \overline{\Phi}_{ik}^{-1} + \frac{(\alpha_i + \alpha_N)\alpha_k}{\alpha_i} \overline{\Phi}_{iN}^{-1} \right) \right. \\
& \left. + (\alpha_i + \alpha_N) \overline{\Phi}_{iN}^{-1} \right] \\
& + \frac{1}{2} \sum_{k=1, k \neq i}^{N-1} \widetilde{c}_k \left((\alpha_i + \alpha_k) \left(1 + \frac{\alpha_N}{\alpha_i} \right) \overline{\Phi}_{ik}^{-1} - (\alpha_k + \alpha_N) \left(\frac{\alpha_i}{\alpha_N} + 1 \right) \overline{\Phi}_{Nk}^{-1} \right. \\
& \left. + (\alpha_i + \alpha_N)\alpha_k \left(\frac{1}{\alpha_N} - \frac{1}{\alpha_i} \right) \overline{\Phi}_{iN}^{-1} \right) \\
& + \frac{1}{2} \sum_{k=1, k \neq i}^{N-1} \left[\sum_{l=1, l \neq i,k}^{N-1} \widetilde{c}_l \left(\frac{\alpha_l (\alpha_k + \alpha_N)}{\alpha_N} \overline{\Phi}_{Nk}^{-1} - \frac{\alpha_l (\alpha_i + \alpha_k)}{\alpha_i} \overline{\Phi}_{ik}^{-1} + \frac{(\alpha_i + \alpha_l)\alpha_k}{\alpha_i} \overline{\Phi}_{il}^{-1} \right. \right. \\
& \left. \left. - \frac{(\alpha_l + \alpha_N)\alpha_k}{\alpha_N} \overline{\Phi}_{Nl}^{-1} \right) \right]
\end{aligned}
$$

Despite a fairly rich expression, this expression depends only on steady state elasticities, consumption shares and the term $\frac{\widetilde{C}}{\overline{\overline{\mathcal{H}}}} \sum_{k=1}^{N} \left(coef_k^{(iN)} \right)$. Note that:

$$
\begin{aligned}
\sum_{k=1}^{N} coef_k^{(iN)} &= \sum_{k=1}^{N} \left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N} \right) \overline{c}_k \\
&= \sum_{k=1}^{N} \left(\frac{\overline{f}_{ik} \overline{c}_k}{\overline{f}_i} \right) - \sum_{k=1}^{N} \left(\frac{\overline{f}_{Nk} \overline{c}_k}{\overline{f}_N} \right) \\
&= \overline{\mathcal{H}}_i - \overline{\mathcal{H}}_N
\end{aligned}
$$

where $\overline{\mathcal{H}}_l \equiv \frac{\sum_{k=1}^{N} \overline{f}_{lk} \overline{c}_k}{\overline{f}_l}$ for all $l = 1, 2, ..., N$.

and therefore, after some rearranging:

$$
\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N = & \widetilde{C}\frac{\overline{\mathcal{H}}_i - \overline{\mathcal{H}}_N}{\overline{\mathcal{H}}} \\
& + \frac{1}{2}\sum_{k=1}^{N} \widetilde{c}_k \sum_{l=1, l\neq k}^{N} \left(\alpha_k \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{il}^{-1} \right) + \alpha_l \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) \right. \\
& \left. + \frac{\alpha_k \alpha_l}{\alpha_N} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) + \frac{\alpha_k \alpha_l}{\alpha_i} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{il}^{-1} \right) \right)
\end{aligned}
$$

with the convention that $\Phi_{xy} = 0$ if $y = x$.

Recalling that $\widetilde{C}$ can be expressed as a function of the consumptions $\widetilde{c}_l$, the steady state ratio $\overline{\mathcal{H}}$ and the steady state consumption shares, the equation above defines the linearised demand function as depending only on steady state consumption shares α_l, steady state bilateral elasticities $\overline{\Phi}_{lm}$ and the steady state ratios $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}_l$ ($l = 1, 2, ..., N$; $m = 1, 2, ..., N$).

Considering the above equation for all $i = 1, \dots N-1$, we have prooved that the first-order dynamics of all relative demand functions depend only on the steady state values of the sufficient statistics listed in Theorem 1.

Market Clearing Condition

$$
\begin{aligned}
Y^{(n)} &= \sum_{i=1}^{N} c_n^{(i)} \\
0 &= Y^{(n)} - \sum_{i=1}^{N} c_n^{(i)} \\
&\approx \overline{Y}^{(n)}\widetilde{Y}^{(n)} - \sum_{i=1}^{N} \overline{c}_n^{(i)} \widetilde{c}_n^{(i)} \\
\widetilde{Y}^{(n)} &\approx \sum_{i=1}^{N} \frac{\overline{c}_n^{(i)}}{\overline{Y}^{(n)}} \widetilde{c}_n^{(i)}
\end{aligned}
$$

Recall that we assumed that in steady state $\overline{C}^{(n)} = \overline{p}_n^{(n)} \overline{Y}^{(n)}$, which implies:

$$
\widetilde{Y}^{(n)} \approx \sum_{i=1}^{N} \frac{\overline{p}_n^{(n)} \overline{c}_n^{(i)}}{\overline{C}^{(n)}} \widetilde{c}_n^{(i)}
$$

We are going to further assume that there is bilaterally balanced trade between every country-pair, which means that $p_i^{(n)}c_i^{(n)} = p_n^{(i)}c_n^{(i)}$. Plugging this in, assuming that $p_n^{(i)} = p_n^{(n)}$:

$$
\begin{aligned}
\widetilde{Y}^{(n)} &\approx \sum_{i=1}^{N} \frac{\overline{p}_i^{(n)}\overline{c}_i^{(n)}}{\overline{C}^{(n)}} \widetilde{c}_n^{(i)} \\
&= \sum_{i=1}^{N} \alpha_i^{(n)} \widetilde{c}_n^{(i)}
\end{aligned}
$$

which again only depends on the $\alpha_i^{(n)}$. □

B Proof of Corollary 1

As shown in Appendix A, the linearised equations characterising the aggregate consumption, the consumer price index and the market clearing conditions already depend only on steady state consumption shares, steady state bilateral elasticities and the steady state ratio $\overline{\mathcal{H}}$. Recall now that the linearised relative demand function equations are defined for all $i = 1, 2, ..., N$ as:

$$
\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N =& \widetilde{C} \frac{\overline{\mathcal{H}}_i - \overline{\mathcal{H}}_N}{\overline{\mathcal{H}}} \\
&+ \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}_k \sum_{l=1, l \neq k}^{N} \left(\alpha_k \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{il}^{-1} \right) + \alpha_l \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) \right. \\
&\left. + \frac{\alpha_k \alpha_l}{\alpha_N} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) + \frac{\alpha_k \alpha_l}{\alpha_i} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{il}^{-1} \right) \right)
\end{aligned}
$$

It is easy to check that a function f homogeneous of degree r has the following property:

$$
\frac{\sum_{k=1}^{N} f_{ik} c_k}{f_i(\mathbf{c})} = r - 1
$$

This implies that $\overline{\mathcal{H}}_i = r - 1$ for all $i = 1, 2, ..., N$. Hence the first term in the linearised relative demand functions is equal to zero, and the steady state consumption shares, bilateral elasticities and the ratio $\overline{\mathcal{H}}$ are sufficient to characterise the dynamics of the model at first order.

$$
\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N =& \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}_k \sum_{l=1, l \neq k}^{N} \left(\alpha_k \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{il}^{-1} \right) + \alpha_l \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) \right. \\
&\left. + \frac{\alpha_k \alpha_l}{\alpha_N} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) + \frac{\alpha_k \alpha_l}{\alpha_i} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{il}^{-1} \right) \right)
\end{aligned}
$$

□

C Kimball Aggregator Derivations

In this Appendix, we derive the elasticity of substitution between two goods i and j, for $i, j = 1, 2, ..., N$ and $i \neq j$, for a a representative consumer in country n, where $n = 1, 2, ..., N$, implied by the Kimball (1995) aggregator. For readability, we drop the country (n) superscripts and the time subscripts. The elasticity of substitution that we derive is defined as:

$$\Phi_{ij} = \frac{\mathrm{d}(c_j/c_i)}{\mathrm{d}(p_i/p_j)} \frac{c_i p_i}{c_j p_j}$$

We note that the first term on the right-hand side of this expression can be written as:

$$\begin{aligned} \frac{\mathrm{d}(c_j/c_i)}{\mathrm{d}(p_i/p_j)} &= \left[\frac{\mathrm{d}(p_i/p_j)}{\mathrm{d}(c_j/c_i)}\right]^{-1} \\ &= \left[\frac{\mathrm{d}(p_i/p_j)}{\mathrm{d}c_j} \frac{\mathrm{d}c_j}{\mathrm{d}(c_j/c_i)}\right]^{-1} \end{aligned} \tag{22}$$

We derive this term in two steps.

First, we solve for the final term in equation (22), which can be expressed as:

$$\begin{aligned} \frac{\mathrm{d}c_j}{\mathrm{d}(c_j/c_i)} &= \left[\frac{\mathrm{d}(c_j/c_i)}{\mathrm{d}c_j}\right]^{-1} \\ &= \left[\frac{\partial(c_j/c_i)}{\partial c_j} + \frac{\partial(c_j/c_i)}{\partial c_i} \frac{\mathrm{d}c_i}{\mathrm{d}c_j}\right]^{-1} \end{aligned}$$

Within this, we can solve for $\frac{\mathrm{d}c_i}{\mathrm{d}c_j}$ by using the total derivative of the aggregator function $C = f(\mathbf{c})$, where $\mathrm{d}C = 0$ and $\mathrm{d}c_k$ for all $k = 1, 2, ..., N$ where $k \neq i, j$. This yields:

$$\frac{\mathrm{d}c_i}{\mathrm{d}c_j} = -\frac{f_j}{f_i} = -\frac{p_j}{p_i}$$

So then:

$$\frac{\mathrm{d}c_j}{\mathrm{d}(c_j/c_i)} = \left[\frac{1}{c_i}\left(1 + \frac{p_j c_j}{p_i c_i}\right)\right]^{-1}$$

Second, we solve for the first term in equation (22). To do this, we note that the relative demand function can be expressed as:

$$\frac{p_i}{p_j} = \frac{\Upsilon'\left(\frac{c_i}{b_i C}\right)}{\Upsilon'\left(\frac{c_j}{b_j C}\right)} \equiv h\left(c_i, c_j, C\right)$$

So then, when $\mathrm{d}C = 0$:

$$
\begin{aligned}
\frac{\mathrm{d}(p_i/p_j)}{\mathrm{d}c_j} =& \frac{\mathrm{d}h}{\mathrm{d}c_j} \\
=& \frac{\partial h}{\partial c_j} + \frac{\partial h}{\partial c_i}\frac{\mathrm{d}c_i}{\mathrm{d}c_j} \\
=& \frac{1}{\sigma c_i}\left(\frac{c_i}{b_i C}\right)^{\frac{\epsilon}{\sigma}} + \frac{p_i}{p_j}\frac{1}{\sigma c_j}\left(\frac{c_j}{b_j C}\right)^{\frac{\epsilon}{\sigma}}
\end{aligned}
$$

Combining the expressions for the first and second terms in equation (22) yields:

$$
\frac{\mathrm{d}(c_j/c_i)}{\mathrm{d}(p_i/p_j)} = \left(\frac{1}{\sigma c_i}\left(\frac{c_i}{b_i C}\right)^{\frac{\epsilon}{\sigma}} + \frac{p_i}{p_j}\frac{1}{\sigma c_j}\left(\frac{c_j}{b_j C}\right)^{\frac{\epsilon}{\sigma}}\right) c_i \left[1 + \frac{p_j c_j}{p_i c_i}\right]^{-1}
$$

With this, the elasticity of substitution can be written as:

$$
\Phi_{ij} = \sigma\left(1 + \frac{\alpha_i}{\alpha_j}\right)\left[\left(\frac{c_i}{b_i C}\right)^{\frac{\epsilon}{\sigma}} + \frac{\alpha_i}{\alpha_j}\left(\frac{c_j}{b_j C}\right)^{\frac{\epsilon}{\sigma}}\right]^{-1} \tag{23}
$$

D Nested-CES Derivations

Here we derive the elasticity of substitution between pairs of goods in the 3-country nested CES set-up detailed in the main text. We consider country (H) along all computations here and therefore drop the superscripts (H) and the time subscripts for readability. Let us recall the formula for the direct partial elasticity between goods x and y:

$$
\Phi_{xy}^{-1} = -\left(\frac{1}{c_x f_x} + \frac{1}{c_y f_y}\right)^{-1}\left[\left(\frac{f_{xx}}{f_x^2} - \frac{f_{xy}}{f_x f_y}\right) + \left(\frac{f_{yy}}{f_y^2} - \frac{f_{xy}}{f_x f_y}\right)\right]
$$

We can apply it to the 3-country nested CES aggregator defined by :

$$
\begin{aligned}
C = f\left(c_H, c_F, c_R\right) = &\left(a_H{}^{\frac{1}{\phi_H}} c_H{}^{\frac{\phi_H - 1}{\phi_H}} + (1 - a_H)^{\frac{1}{\phi_H}} C_{FR}{}^{\frac{\phi_H - 1}{\phi_H}}\right)^{\frac{\phi_H}{\phi_H - 1}} \\
\text{where} \qquad C_{FR} = &\left(a_F{}^{\frac{1}{\phi_F}} c_F{}^{\frac{\phi_F - 1}{\phi_F}} + (1 - a_F)^{\frac{1}{\phi_F}} C_R{}^{\frac{\phi_F - 1}{\phi_F}}\right)^{\frac{\phi_F}{\phi_F - 1}}
\end{aligned}
$$

First, we compute the partial derivatives of f.

$$
\begin{aligned}
f_H &= a_H^{\frac{1}{\phi_H}} \left(\frac{C}{c_H}\right)^{\frac{1}{\phi_H}} \\
f_F &= (1-a_H)^{\frac{1}{\phi_H}} a_F^{\frac{1}{\phi_F}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}} \left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \\
f_R &= (1-a_H)^{\frac{1}{\phi_H}} (1-a_F)^{\frac{1}{\phi_F}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}} \\
f_{HH} &= \frac{1}{\phi_H} f_H \left(-\frac{1}{c_H} + \frac{1}{C} f_H\right) \\
f_{HF} &= f_{FH} = \frac{1}{\phi_H} a_H^{\frac{1}{\phi_H}} (1-a_H)^{\frac{1}{\phi_H}} a_F^{\frac{1}{\phi_F}} \left(\frac{C}{c_H}\right)^{\frac{1}{\phi_H}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}} \left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C} = \frac{1}{\phi_H C} f_H f_F \\
f_{HR} &= \frac{1}{\phi_H C} f_H f_R \\
f_{FF} &= f_F \left(-\frac{1}{\phi_F c_F} + \frac{\phi_H - \phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}} + \frac{1}{\phi_H C} f_F\right) \\
f_{FR} &= f_F (1-a_F)^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}} \left(\frac{\phi_H - \phi_F}{\phi_H \phi_F} \frac{1}{C_{FR}} + \frac{1}{\phi_H C} (1-a_H)^{\frac{1}{\phi_H}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}}\right) \\
&= f_F \left(\frac{1}{\phi_H C} f_R + \frac{\phi_H - \phi_F}{\phi_H \phi_F} \frac{1}{C_{FR}} (1-\alpha_F)^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}}\right) \\
f_{RR} &= f_R \left(-\frac{1}{\phi_F c_R} + \frac{\phi_H - \phi_F}{\phi_H \phi_F} (1-a_F)^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}} + \frac{1}{\phi_H C} f_R\right)
\end{aligned}
$$

Using the above, we compute the inverse of the bilateral elasticities.

$$
\begin{aligned}
\Phi_{HF}^{-1} =& -\left(\frac{1}{c_H f_H}+\frac{1}{c_F f_F}\right)^{-1}\left[\left(\frac{f_{HH}}{f_H^2}-\frac{f_{HF}}{f_H f_F}\right)+\left(\frac{f_{FF}}{f_F^2}-\frac{f_{HF}}{f_H f_F}\right)\right] \\
=& -\left(\frac{1}{x_H f_H}+\frac{1}{x_F f_F}\right)^{-1} \\
&\times\left[\left(\frac{\frac{1}{\phi_H} f_H\left(-\frac{1}{c_H}+\frac{1}{C} f_H\right)}{f_H^2}-\frac{\frac{1}{\phi_H C} f_H f_F}{f_H f_F}\right)\right. \\
&\left.+\left(\frac{f_F\left(-\frac{1}{\phi_F c_F}+\frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{1}{C_{FR}}+\frac{1}{\phi_H C} f_F\right)}{f_F^2}-\frac{\frac{1}{\phi_H C} f_H f_F}{f_H f_F}\right)\right] \\
=& -\left(\frac{1}{c_H f_H}+\frac{1}{c_F f_F}\right)^{-1} \\
&\times\left[\left(\frac{\left(-\frac{1}{\phi_H c_H}+\frac{1}{\phi_H C} f_H\right)}{f_H}-\frac{1}{\phi_H C}\right)\right. \\
&\left.+\left(\frac{\left(-\frac{1}{\phi_F c_F}+\frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{1}{C_{FR}}+\frac{1}{\phi_H C} f_F\right)}{f_F}-\frac{1}{\phi_H C}\right)\right] \\
=& -\left(\frac{c_F f_F+c_H f_H}{c_H f_H c_F f_F}\right)^{-1} \\
&\times\left[-\frac{1}{\phi_H c_H f_H}+\frac{1}{\phi_H C}-\frac{1}{\phi_H C}-\frac{1}{\phi_F c_F f_F}+\frac{\phi_H-\phi_F}{\phi_H \phi_F f_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{1}{C_{FR}}+\frac{1}{\phi_H C}-\frac{1}{\phi_H C}\right] \\
=& -\frac{c_H f_H c_F f_F}{c_F f_F+c_H f_H}\times\left[-\frac{1}{\phi_H c_H f_H}-\frac{1}{\phi_F c_F f_F}+\frac{\phi_H-\phi_F}{\phi_H \phi_F f_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{1}{C_{FR}}\right] \\
=& -\left[-\frac{c_F f_F}{c_F f_F+c_H f_H}\frac{1}{\phi_H}-\frac{c_H f_H}{c_F f_F+c_H f_H}\frac{1}{\phi_F}+\frac{c_H f_H}{c_F f_F+c_H f_H}\frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{c_F}{C_{FR}}\right] \\
=& \frac{c_F p_F}{P_{FR} C_{FR}}\left(\frac{c_F p_F+c_H p_H}{P_{FR} C_{FR}}\right)^{-1}\frac{1}{\phi_H}+\frac{c_H p_H}{PC}\frac{PC}{c_F p_F+c_H p_H}\frac{1}{\phi_F} \\
&-\frac{c_H f_H}{c_F f_F+c_H f_H}\frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{c_F}{C_{FR}} \\
=& \frac{c_F p_F}{PC}\frac{PC}{P_{FR} C_{FR}}\left(\frac{c_F p_F+c_H p_H}{PC}\frac{PC}{P_{FR} C_{FR}}\right)^{-1}\frac{1}{\phi_H} \\
&+\frac{c_H p_H}{PC}\frac{PC}{c_F p_F+c_H p_H}\frac{1}{\phi_F}-\frac{c_H f_H}{c_F f_F+c_H f_H}\frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{c_F}{C_{FR}} \\
=& \frac{\alpha_F}{1-\alpha_H}\left(\frac{\alpha_F+\alpha_H}{1-\alpha_H}\right)^{-1}\frac{1}{\phi_H}+\alpha_H\left(\alpha_F+\alpha_H\right)^{-1}\frac{1}{\phi_F}-\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{c_F}{C_{FR}} \\
=& \frac{\alpha_F}{\alpha_F+\alpha_H}\frac{1}{\phi_H}+\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{1}{\phi_F}-\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{\phi_H-\phi_F}{\phi_H \phi_F}\frac{p_F}{P_{FR}}\frac{c_F}{C_{FR}} \\
=& \frac{\alpha_F}{\alpha_F+\alpha_H}\frac{1}{\phi_H}+\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{1}{\phi_F}-\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{\phi_H-\phi_F}{\phi_H \phi_F}\frac{\alpha_F}{1-\alpha_H} \\
=& \frac{1}{\phi_H \phi_F\left(\alpha_F+\alpha_H\right)\left(1-\alpha_H\right)}\left[\alpha_F(1-\alpha_H)\phi_F+\alpha_H(1-\alpha_H)\phi_H-\alpha_H \alpha_F(\phi_H-\phi_F)\right] \\
\Phi_{HF}^{-1} =& \frac{\alpha_F \phi_F+\alpha_H \alpha_R \phi_H}{\phi_H \phi_F\left(\alpha_F+\alpha_H \alpha_R\right)}
\end{aligned}
$$

Hence:

$$\Phi_{HF}^{(H)} = \frac{\phi_H \phi_F \left(\alpha_F^{(H)} + \alpha_H^{(H)} \alpha_R^{(H)}\right)}{\alpha_F^{(H)} \phi_F + \alpha_H^{(H)} \alpha_R^{(H)} \phi_H}$$

We can compute $\Phi_{HR}^{(H)}$ in a similar fashion and obtain symmetrically:

$$\Phi_{HR}^{(H)} = \frac{\phi_H \phi_F (\alpha_R^{(H)} + \alpha_H^{(H)} \alpha_F^{(H)})}{\alpha_R^{(H)} \phi_F + \alpha_H^{(H)} \alpha_F^{(H)} \phi_H}$$

And it is easy to check that $\Phi_{FR}^{(H)} = \phi_F$.