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Abstract

I model the dynamics of price adjustments to news arrival in limit order markets when investors have limited attention. Because of limited attention, investors monitor news arrival imperfectly. Consequently prices reflect news with delay. This delay shrinks when investors' attention capacity increases. The adjustment delay also decreases when the frequency of news arrival increases. When news arrival frequency is higher, the picking-off risk increases for limit orders. The order book becomes thinner and there are fewer stale limit orders to execute or cancel following news arrival. Hence, it reduces the time it takes for market prices to reflect news content.

Keywords: imperfect attention, news, limit order book, price formation, market liquidity.

JEL classification: G14, D82, D83

Résumé

Je modélise la dynamique des ajustements de prix à l'arrivée de nouvelles dans les marchés conduits par les ordres, lorsque les investisseurs ont une capacité d'attention limitée. En raison de leur attention limitée, les investisseurs suivent imparfaitement l'arrivée de nouvelles. Ainsi les prix s'ajustent aux nouvelles après un certain délai. Ce délai diminue lorsque le niveau d'attention des investisseurs augmente. Lorsque la fréquence d'arrivée de nouvelles augmente, le risque de sélection adverse pour les ordres à cours limité est amplifié. Le carnet d'ordre devient plus fin et il y a donc moins d'ordres à cours limité à exécuter ou annuler à la suite de l'arrivée d'une nouvelle. Ainsi les prix reflètent plus rapidement le contenu informationnel des nouvelles.

Mots-clés: marché financier, marché dirigé par les ordres, nouvelles, attention limitée, surveillance de marché imparfaite.

Classification JEL: G14, D82, D83

1 Introduction

Investors have limited attention capacities and thus cannot continuously monitor the flow of information in financial markets. As a result, they are unable to get or analyze instantaneously the implications of public financial news when it arrives. And news content cannot instantaneously turn into common knowledge for the market. Consequently, at the short horizon public information is private information for investors who observe it first. Because of limited attention, the release of public information generates a short-term period of information asymmetry. How do financial markets react around news arrival? And what role does limited attention play in this process?

To address these questions I propose a theoretical framework to analyze the role of limited attention on market reaction to news. I design a model of limit order market in the presence of uncertainty on the asset value due to news arrival. This model extends the OTC markets framework of Duffie, Garleanu and Pedersen (2005, 2007) to limit order markets. In Duffie et al., the main market friction is a search friction for trading counterparties. In my model the market friction is a limited attention friction which prevents investors from perfectly monitoring the market and the arrival of news. Investors cannot continuously observe public information or trade in the market but do it at random times. This set-up generates a gradual dissemination of new public information among investors following news arrival. Introducing imperfect market monitoring by investors allows to jointly describe market liquidity, price formation and market efficiency around news arrival.

Financial markets' reactions to public information releases has motivated an extensive line of research both empirical and theoretical especially since the 1990's. This literature aims to understand the price formation process around these events. This issue has been addressed empirically by several papers, such as Eredington and Lee (1995), Fleming and Remolona (1999) and Green (2004). These papers study the reaction of US Treasury securities markets to scheduled macroeconomic announcements. The first two papers show that the market reacts to the announcement in two successive phases. In the first phase, the price shifts quickly to a new level in line with the main elements of the announcement. The second phase of this reaction is characterized by high volatility, suggesting that investors disagree on the precise interpretation of the announcement. This phase ends when the interpretations of market participants eventually converge. Green's paper shows that these macro announcements increase the level of adverse selection, suggesting that investors with better processing abilities can take advantage of these events.

With respect to this literature, a significant contribution of this paper is to design a model that considers unscheduled news. Financial news is released every day by news providers, such as Thomson Reuters and Bloomberg, and delivers relevant information for valuing asset prices. Virtually all this news arrives at unscheduled times[1]. In addition, the frequency of these news arrival events varies a lot across stocks[2]. The unscheduled nature of these events is likely to prevent investors from paying perfect attention to financial news. By embedding limited attention in a limit order book model, I can address the question as to how unscheduled news affects trading decisions and ultimately price formation and liquidity provision.

Recent developments in the structure of financial markets suggest that imperfect market monitoring does indeed play an important role in market reactions to news releases. In recent years, there has been a tremendous growth in High Frequency Trading activities that specialize in trading very fast on financial news, which requires the use of intensive monitoring technologies. This boom illustrates the gains that improving market monitoring capacities can yield, for some market participants. Ultimately, this trend shows how important attention capacity, allocated to market monitoring, can be for any investor, not only to keep up with the fastest market players but also to optimize their trading strategies[3]. Obviously, these technological developments have been enabled and fostered by the modernization of financial markets architecture. Now most equities and derivatives are traded in centralized electronic exchanges organized as limit order markets. The way trading is organized is not immaterial to understanding the effect of the imperfect monitoring of news. That is why, in my model, I explicitly look at limit order markets.

The paper has several empirical implications for market liquidity and price dynamics around news arrival. When the frequency of news arrival increases, (i) the level of market liquidity, measured by market depth, decreases, (ii) prices adjust faster following the arrival of news and (iii) the number of limit order cancellations in the price adjustment process declines relative to limit order executions. The intuition for these results stems from the short-

[1] In a sub-sample of 40 large stocks, representing 70% of the market cap of the FTSE100, Gross-Klussmann and Hautsch (2011) find that one stock receives on average 750 pieces of unscheduled news over 1.5 years

[2] In the same sub-sample, the news arrival frequency varies ten-fold, from 200 to 2000 pieces of news

[3] Some brokerage firms use a smart order routing algorithm to execute their clients' orders. These programs monitor markets to evaluate optimal execution conditions for these orders.

term period of information asymmetry around news arrival that is due to limited attention. Consistent with the presence of information asymmetry, there is a "picking-off" risk for liquidity suppliers and this risk varies with the frequency of news arrival. Investors may be reluctant to supply liquidity with limit orders since, following news arrival, limited attention delays their reaction. In the meantime, their limit orders can be "picked-off" because the price at which they are posted is not in line with the new asset value, and they therefore offer a profit opportunity.

In the model, investors can both supply liquidity with limit orders and consume liquidity with market orders. Before news arrival, investors trade with each other because they have different private values for the asset, which generates gains from trade. During this phase, the limit order book remains in a steady state. The market depth (the number of limit orders in the order book) is constant and determined by the following trade-off. Market orders provide execution immediacy whereas limit orders provide price improvement but bear execution delay and a picking-off risk when the asset value changes. At equilibrium, market depth adjusts so that investors are indifferent between market and limit orders.

When, following news arrival, the asset value changes, this is publicly available but investors do not observe the change immediately. They become aware of it with a time lag that depends on their market monitoring intensity. This generates a transition phase at the end of which prices adjust to the new asset value. This price formation process relies on two underlying dynamics. First, investors who observe the new asset value fast enough can profit from a temporary arbitrage opportunity by using market orders to "pick-off" stale limit orders at the initial price. Second, investors with limit orders in the order book cancel these orders to avoid being picked-off by previous market orders. Once limit orders at the initial price level have all been cancelled or picked-off, the transition phase ends and the limit order book converges to a new steady state, with no future news. Thus, the model provides a high-frequency description of price and order dynamics around news arrival. This should prove useful for empiricists[4].

The decision for investors to use limit or market orders to trade before news arrival depends on the risk of being picked off during the transition phase. All other things being equal, this risk increases the expected loss associated with limit order submission and has a

[4]Engle et al. (2009) use high-frequency limit order book data to analyze liquidity and volatility in the U.S. Treasury market.

negative impact on market depth. In this context, the effect of the frequency of news arrival is intuitive. More frequent news arrivals increase the likelihood of an event after which limit orders may be picked off, which leads to higher picking-off risk. Consequently market depth depends negatively on this frequency. Since market depth is thinner, the amount of stale limit orders that must be cancelled or executed in the transition phase is smaller, which makes the price adjustment faster. Finally, during the transition phase, the flow of limit order cancellations is proportional to market depth and depends negatively on news arrival frequency, while the flow of limit order executions corresponds to the flow of market orders, which does not depend on the news arrival frequency. This is why the number of limit order cancellations decreases relative to the number of executions when news arrival frequency increases.

Investors' limited attention capacity also influences the level of picking-off risk prior to news arrival. However, it has an ambiguous effect. To see why, let us consider an increase of investors' monitoring intensity[5]. On the one hand, investors can cancel their limit orders faster after news arrival, which reduces their risk of being picked off and makes limit orders more profitable. However, investors can also send market orders faster to execute against stale limit orders, following news arrival, which aggravates the risk of being picked off for limit orders. Overall, limit orders may be more or less profitable after an increase in monitoring intensity. I identify conditions under which limit orders are overall more profitable after such an increase. However the magnitude of its effect on market depth appears to be very small, especially if compared with the effect that news arrival frequency has. This suggests that monitoring abilities can have a quantitative impact on investors' trading strategies only when they are different between investors. Only relative monitoring abilities matter.

The paper is organized as follows. Section 2 reviews the related literature. Section 3 presents the set-up and assumptions of the model. Section 4 gives the equilibrium and its general description. Sections 5, 6 and 7 describe the properties of the different phases involved in the equilibrium dynamics of the limit order market. Section 8 discusses empirical implications of the model. Section 9 concludes.

[5]This increase may result from a reduction of latencies.

2 Literature review

A number of theoretical works have studied the reaction of financial markets to public information releases. For instance, Kim and Verrecchia (1994) propose a model to analyze the market reaction to earnings announcements. They assume that traders have heterogenous information processing abilities, which advantages the skilled ones. They then make empirical predictions about market liquidity and trading volumes around these events. The essential difference between a model *à la* Kim and Verrecchia and the model in this paper is that the former does not consider a realistic trading mechanism and cannot make predictions about the dynamics of order flows. As mentioned in section 1 several empirical studies have analyzed market reactions to public information, such as Eredington and Lee (1993, 1995), Fleming and Remolona (1999) and Green (2004). More recently Tetlock (2010) uses a large data set on all types of public news to address the effect of public news releases on the level of asymmetric information and how it affects returns around these events. Della Vigna and Pollet (2009) link market reactions to earnings announcements and investors' attention to explain post-earnings announcement drift.

The link between attention capacity and investors' decision-making in financial markets is a fairly new research topic. Recent studies by Peng and Xiong (2006), Van Nieuwerburgh and Veldkamp (2009) and Mondria (2010) have developed theories in which investors have limited attention capacity and must allocate it across different assets. The more they allocate attention to an asset the more precise their information about its future payoff is. Through this channel, these papers analyze the effect of limited attention on portfolio diversification and asset prices. My paper contributes to this literature by addressing limited attention from a different angle. It maps limited attention capacity to imperfect market monitoring at the high-frequency level, which allows me to analyze its effect on the optimal order choice for traders to implement their trading strategies.

In his presidential address (2010), Darrell Duffie stresses that market monitoring imperfection is a key determinant to understanding market dynamics. Foucault, Kadan and Kandel (2013) address this problem in a limit order book setting. They consider traders with heterogenous private values for the asset who can post orders to trade with each other and can choose ex ante their frequency of market monitoring. But they are exogenously defined as limit order or market order users. Biais, Hombert and Weill (2013) also consider agents with imperfect market monitoring abilities. The structure of their model has similarities with the one proposed here. However, the focus of their papers is different since they consider limit order market dynamics generated by aggregate liquidity shocks rather than by a change in the asset value. Moreover I model how asset value uncertainty affects trading strategies before this change occurs. In Biais et al., market dynamics start with a liquidity shock. Pagnotta and Philippon (2013) study the effect of competition between exchanges on the market monitoring intensity, or latency, they provide to their customers. They show that competition is an incentive for investing in fast trading technologies. Biais et al. (2013), Pagnotta and Philippon (2013), as well as my paper, use and adapt the model of search friction in OTC markets introduced by Duffie, Gârleanu and Pedersen (2005, 2007) and extended by Lagos and Rocheteau (2009), Lagos, Rocheteau and Weill (2011), Vayanos and Weill(2008) and Weill (2007,2008).

The effect of information monitoring on market liquidity has been studied by Foucault, Roëll and Sandas (2003) in the case of a dealership market. In their model, market makers face adverse selection due to the presence of informed traders. They can reduce the adverse selection risk by monitoring information which allows them to acquire information before the informed traders and, thus, to adjust their quote accordingly. The choice of intense monitoring is costly. In my model monitoring intensity is an exogenous parameter but it affects both liquidity suppliers and demanders, which is consistent with how limit order markets work. Goettler, Parlour and Rajan (2009) design a very realistic environment of limit order market that is not tractable and meant to be solved numerically. In their paper, traders do not continuously monitor the market and decide ex ante to be privately informed or not about the asset value.

This paper also builds on the dynamic limit order market literature. This literature is quite scarce when compared to its practical importance. One of the reasons for this is that limit order markets are very hard to model. Foucault (1999) and Parlour (1998) are the first models of limit order markets designed as dynamic games that deal with the inter-temporal aspect of the problem. The tractability of these models is appreciable but is reached at the cost of strong assumptions. Both incorporate private and/or common value as drivers of trading and price formation processes but do not allow for strategic decisions (cancellation, resubmission) over the limit order lifetime. Foucault, Kadan and Kandel (2005) focus on the dynamics of liquidity supply in a limit order market. In their paper, investors trade for liquidity reasons. They have different preferences for immediacy, which determines their order choice. Rosu (2009) generalizes Foucault, Kadan and Kandel (2005) framework and designs a continuous time model in which traders can freely send limit orders at any price and can cancel them. Rosu (2013) adds a varying asset common value to his previous model. The two papers by Rosu are based on the fundamental assumption that limit orders are continuously monitored by their owners. As in Rosu's models, I design a framework that allows for complete freedom of choice for investors' order management, with the exception of the zero or one unit holding constraint (as in Rosu (2009,2013)). Pagnotta (2013) designs a limit order book model with insider trading in which agents optimally choose their trading frequency. At equilibrium they do not trade continuously but they continuously observe the market and revise their beliefs accordingly.

3 Model

3.1 Preferences and asset dynamics

I consider a continuous time framework with an infinite horizon, $t \in [0, +\infty)$. There is a continuum of investors $[0, 1]$. They are risk neutral and infinitely lived, with time preferences determined by a time discount rate $r > 0$. These investors can trade an asset that has a value v_t that changes over time.

Preferences. As in Duffie, Gârleanu and Pedersen (2005,2007), an investor is characterized by an intrinsic private value for the asset that can be "high" or "low". An investor with a high private value receives a utility flow v_t per asset unit that she owns. An investor with a low private value receives a utility flow $v_t - \delta$ per asset unit that she owns. Between time t and time $t + dt$ the private value of an investor can switch from one level to another (high to low or low to high) with probability $\rho.dt$.

Asset holding and supply. As in Duffie et al., I assume that an investor can own either one or zero unit of the asset - this is the *"0 or 1" asset holding constraint*. The asset supply is equal to $\frac{1}{2}$. These two assumptions imply that half of the population owns the asset[6].

[6]The choice of a supply of 1/2 allows to make the problem symmetric between buyers and sellers of the asset. The results of the paper should hold, by a continuity argument, if the supply is equal to $s \in [0, 1]$ with s "close enough" to 1/2.

Types of investors. In the remainder of the paper we define the type of each investor as a combination of her asset holding status and her private value for the asset. The type of each investor belongs to the set $\{ho, hn, lo, ln\}$ (h: high, l: low, o: owner, n: non-owner). Hence, we can divide the mass of investors into 4 populations: L_{ho}, L_{hn}, L_{lo}, L_{ln}. Given the assumptions made about investors' preferences and holding constraints, they must verify the following equations

$$L_{ho} + L_{hn} + L_{lo} + L_{ln} = 1, \tag{1}$$

$$L_{ho} + L_{lo} = \frac{1}{2}. \tag{2}$$

It is possible to extend the number of possible types by taking into account the limit order submission status of investors. Indeed, in a limit order book, an owner can either be out of the market or have an order in the order book. This is also true for a non-owner. This setting can generate many sub-types of the previous types. Let us call $\mathcal{T}$ the set of all possible types. If an investor does not have any limit order submitted in the order book, she is *out*. If she has a limit order submitted, we have to specify at what price. For instance a ln type can be $ln - out$ or $ln - B$ with a buy limit order at price B. Symmetrically a lo type can be $lo - out$ or $lo - A$ with a sell limit order at price A.

Asset value dynamics. The dynamics of the asset common value v_t are the following:

- at $t = 0$, the asset common value is equal to v_0
- at date $\tau > t$, the common value of the asset changes. This time τ is random and follows a Poisson distribution of intensity μ, $\mathcal{P}(\mu)$. At date τ the common value switches to $v_u = v_0 + \omega$ or $v_d = v_0 - \omega$ with equal probabilities $\frac{1}{2}$
- for $t > \tau$ the asset value is v_u or v_d until the end of the game.
- for $0 < t < \tau$ the state of the world is $\zeta = 0$. For $\tau < t$ the state of the world is either $\zeta = u$ if $v_t = v_0 + \omega$ or $\zeta = d$ if $v_t = v_0 - \omega$.

In this set-up, time τ corresponds to the news arrival event. Consequently, μ can be interpreted as the news arrival frequency since it is the likelihood for such an event to occur in the next period. Parameter ω measures the news surprise, that is, the innovation in the asset common value that is publicly released by the piece of news.

Assumption 1.

$$\omega > 3\delta \times \max\left[1, \frac{2r+\rho}{2\rho}\right]. \tag{3}$$

Assumption 1 states that ω is big compared to δ. It ensures that, if the change in the asset common value, with magnitude ω, is not followed by a change in price, the profit opportunity, measured by ω, is bigger than gains from trade due to differences in private values, measured by δ.

The asset value is seen here as a utility flow for the investor who holds one share. It is directly taken from Duffie et al. However, one could argue that this way of modelling the asset value implicitly considers time horizons that are longer than that which is relevant for this paper (i.e. one trading day). I argue here that this is not a concern since the former model is equivalent to other dynamics for which the asset pays off at some random time and does not provide a continuous flow of utility. For instance, the following formulation yields the same result; the asset pays off cash-flow $V = \frac{v_t}{r}$ at a random time that occurs with respect to a Poisson process of intensity r, and the low private value corresponds to a cost of holding the asset, which is equal to δ per unit of time.

3.2 Limited attention

News arrival can have an interesting effect on financial markets only if these markets are not fully efficient. News is public information and would be immediately impounded into prices otherwise. Hence, we need to introduce a friction that would prevent market participants from having a "perfect" reaction following news arrival. In this paper, I consider investors who monitor the market imperfectly, in the sense that I explain in assumption 2 below.

Conceptually, imperfect market monitoring can stem from investors' limited attention or, put another way, from their limited cognitive capacity. This simply means that a human being is not able to process information instantaneously - he needs time. Moreover, he cannot process more than a limited number of information sources at the same time. He must decide which sources to allocate his attention capacity to. For instance, an investor will not pay attention all day to a particular market. As a consequence, market monitoring is imperfect: the investor cannot track and interpret continuously the flow of public news and other market activities. Neither is he able to trade at all moments during the day.

In the model, I exogenously define the imperfect market monitoring process of investors. Of course, we may think that this process should be endogenous and depend on the optimal allocation of attention capacity by investors. This problem constitutes a natural extension of the model introduced here but is not treated in this paper.

Assumption 2. *I assume that an investor observes the asset value, the market, her private value and contacts the market at random times* $\{t_i\}_{i\in\mathbb{N}}$*. I call these times "market monitoring times". This sequence of market monitoring times is generated by a Poisson process of intensity* $\lambda + \rho$*. More specifically, between time* t *and* $t + dt$*, an investor monitors the market in two situations:*

- *when she uses market monitoring technology, which occurs with probability* $\lambda.dt$
- *when her private value changes, which occurs with probability* $\rho.dt$*.*

The level of investors' attention allocated to market monitoring is measured by λ. At the limit $\lambda = \infty$ investors monitor the market continuously. In a sense, they are infinitely attentive to the market and to the flow of news.

The second point of assumption 2 states that investors continuously monitor their private value for the asset and contact the market whenever this private value changes. This assumption makes it possible to reduce the anticipation problem of the investor who has to take into account the possibility of future shocks to her private value especially when facing the decision to send a limit order. Indeed, she knows that when a shock occurs she has the possibility of cancelling a previous limit order. This prevents her from being executed when it is no longer optimal given her new private value.

3.3 The limit order market

Trading takes place in a limit order market. Prices at which trades can occur belong to a discrete set of prices, the price grid. The minimum difference between two prices is the tick size, Δ. Investors can use limit or market orders to trade. Limit orders are orders that specify a limit price at which the order can be executed. They are posted in the order book until matched with a market order. The depth of the limit order book at price P, D_P, is the number of limit orders posted at price P. Market orders do not specify a price limit. They hit the most competitive limit orders and are immediately executed.

Assumption 3. *There is an integer $K \in \mathbb{N}$ such that the price grid is the following set $\mathcal{G}$,*

$$\mathcal{G} = \{k\Delta,\ k = 0, 1, 2, \dots K\}.$$

There are three integers $0 < k_d < k_0 < k_u < K$ such that

$$\text{for } \ * = 0, u \text{ or } d, \ \frac{v_*}{r} - \frac{\delta}{2r} = \left(k_* + \frac{1}{2}\right)\Delta. \tag{4}$$

The assumption that the price grid is bounded is made for technical reasons. It is reasonable since trading will not occur at prices higher than a certain threshold since the asset value is bounded (the corresponding strategies would be strictly dominated by a strategy in which investors do not trade). A similar assumption is made in Parlour (1998), Foucault, Kadan and Kandel (2005) for instance. The second point of assumption 3 will impose that the average asset private value $\frac{v_*}{r} - \frac{\delta}{2r}$ is at the same distance from the closest bid and ask prices, which facilitates the solving of the equilibrium.

Assumption 4. *In the limit order book, limit orders are executed following a "pro-rata matching" execution rule. In this set-up, all limit orders submitted at the same price have the same probability of execution at any point in time, regardless of their submission date.*

Assumption 4 is a simplifying assumption because two investors of the same type and with a limit order in the book at the same price will have the same value function. In almost all stock limit order markets, the rule is different since the time priority applies. However, in practice, there are some limit order markets where "pro-rata matching" is implemented, in particular in some Euribor or Eurodollar futures markets, studied in Field and Large (2012). Moreover, if we think of the present model as the order book consolidated across multiple trading platforms then the reality of the time priority is mitigated. Indeed, the flow of market orders is split among different trading platforms. Then a limit order in one platform may be executed before a limit order sent previously on another platform.

Assumption 5.

$$\delta > (r + 4\rho)\Delta \tag{5}$$

Assumption 5 states that $\frac{\delta}{r}$ is big compared to Δ. It ensures that the gains from trade due to differences in private values, measured by $\frac{\delta}{r}$, is bigger that the implicit cost of trading, the bid-ask spread, which is measured by the tick size, Δ.

3.4 Value function and equilibrium concept

To finalize the model's description, I first define an investor's action set and strategy in this game. Based on this, I can formally define the value function of an investor and the equilibrium concept that I use to solve the model.

Action set. Each time an investor monitors the market at a "market monitoring time" she can decide to take some actions. These actions are a combination of the following "elementary actions".

- As an owner, she can take the following "elementary actions"; a_o^1: do nothing and remain an owner; a_o^2: submit a sell limit order; a_o^3: submit a sell market order and become a non-owner; a_o^4: cancel her previous sell limit order.
- As a non-owner, she can take the following "elementary actions"; a_n^1: do nothing and remain a non-owner; a_n^2: submit a buy limit order; a_n^3: send a buy market order and become an owner; a_n^4: cancel her previous buy limit order.

Assumption 6. *Element a of action set $\mathcal{A}$ is a sequence of the "elementary actions" defined above, such that, after this sequence is implemented by an investor, the "0 or 1" asset holding constraint is satisfied and the investor has at most 1 limit order in the book, the "0 or 1" order status constraint.*

$$\mathcal{A} = \{a \in \{a_o^1, a_o^2, a_o^3, a_o^4, a_n^1, a_n^2, a_n^3, a_n^4\}^n, n \in \mathbb{N},\ \textit{s.t a satisfies the "0 or 1" constraints}\}.$$

The definition of the action set implies that the investor is not constrained in her actions, except that she can manage at most one order in the book so we are sure that she cannot have two limit orders executed which would violate the "0 or 1" asset holding constraint. Otherwise it is possible for an investor to submit a sequence of buy and sell limit orders an indeterminate number of times, or to submit and cancel a limit order many times during the same "market monitoring time". However, it is obvious that the former action will not be undertaken at equilibrium since implementing these multiple round-trips only leads to paying the bid-ask spread multiple times. In the second example as well, cancelling and submitting does serve the purpose of an investor. If she wants to submit a market order in the end, generating this "flickering quote" first cannot be a way of manipulating the market since the investor's impact is infinitesimal, hence submitting the final order of the sequence straightaway is totally equivalent. More generally, we can intuitively see that at each "market monitoring time" an investor will not trade more than once, otherwise, given the "0 or 1" asset holding constraint, it would involve round-trips that are a strictly dominated strategy. And if the action ends with a limit order submission, there will be at most one limit order cancellation before, because making numerous prior submissions and cancellations is pointless.

Strategy. At a "market monitoring time", an investor takes an action based on strategy. An investor's strategy is a function, σ, that maps the history, her current type, the current state of the market and the actual time to the action set,

$$\sigma : \mathcal{H} \times \mathcal{Q} \times [0, \infty) \to \mathcal{A},$$
$$(h, q, t) \mapsto a.$$

Set $\mathcal{Q}$ brings together all the potential state variables. An element of this set $q \in \mathcal{Q}$ is defined as $q = (\theta, v, S)$ where $\theta \in \mathcal{T}$ is the type of the investor, v the common value of the asset and S the aggregate state of the limit order book, that is, the bid and ask prices and all of the market depths at these prices. $\mathcal{H}$ is the set of all the possible histories of an investor's actions and observations:

$$\mathcal{H} = \{h \in (a_{t_1}, \ldots, a_{t_n}, q_{t_1}, \ldots, q_{t_n}, t_1, \ldots, t_n) \in \mathcal{A}^n \times \mathcal{Q}^n \times [0, \infty)^n, t_1 < \ldots < t_n, n \in \mathbb{N}\}.$$

Value function. Given an investor's strategy, σ, and the set of strategies of all the other investors, noted Σ, we can define her asset holding process $\eta_t \in \{0, 1\}$ that is equal to 1 when she holds one unit of the asset and 0 otherwise. We call her type process $\theta_t \in \mathcal{T}$. And finally, we note P_t the process of trading prices at which her orders are executed, that is, at times when η_t switches from 0 to 1 or vice versa. Hence, at each time t the value function of an investor with strategy σ can be written as:

$$V(h_t, q_t, t; \sigma, \Sigma) = \mathbb{E}_t \int_t^{\infty} e^{-r(s-t)} [\eta_s (v_s - \delta \mathbb{I}_{\{\theta_s \in \, lo\}}) ds - P_s d\eta_s]. \tag{6}$$

Equilibrium concept. The strategy σ is a best response to the other players' set of strategies Σ if and only if for any strategy γ,

$$\forall h_t \, \forall q_t \, \forall t \quad V(h_t, q_t, t; \sigma, \Sigma) \geq V(h_t, q_t, t; \gamma, \Sigma). \tag{7}$$

In this paper, I focus on *Markov perfect equilibria*, in which the strategies depend only on state variables, $q = (\theta, v, S)$. In these type of equilibria, investors make decisions based on their current type θ, on the current set of the public information, that is, on the current asset value v, and on the current aggregate state of the market S.

4 Equilibrium

In the first part of this section, I set out the equilibrium result of the paper. I focus on the symmetric equilibrium. This equilibrium constitutes the core of the paper. I describe it in more detail in sections 5, 6 and 7. I derive empirical implications from the equilibrium result in section 8. In the second part of this section, I show that this is not a unique equilibrium.

4.1 The symmetric equilibrium

Proposition 1. *There is an equilibrium which involves three pair of bid and ask prices. Each pair of prices is associated with one possible asset common value, v_0, v_u or v_d: its bid and ask prices are symmetrical with respect to the average private value of the asset, $\frac{v_*}{r} - \frac{\delta}{2r}$ ($* = 0,\ u$ or d),*

$$B^* = \frac{v_*}{r} - \frac{\delta}{2r} - \frac{\Delta}{2}, \quad A^* = \frac{v_*}{r} - \frac{\delta}{2r} + \frac{\Delta}{2}. \tag{8}$$

(A^0, B^0) is the pair of bid and ask prices at the beginning of the game. (A^u, B^u) (resp. (A^d, B^d)) is the pair of bid and ask prices at the end of the game if the asset common value jumps to v_u (resp. v_d).

Before news arrival, the depths of the limit order book at prices A^0 and B^0 are both equal to the depth parameter α^0 ($D_{A^0} = D_{B^0} = \alpha^0$). Limit orders posted at these prices are executed at idiosyncratic random times, according to a Poisson distribution of intensity l^0. The two parameters, α^0 and l^0, are interdependent:

- *l^0 is such that investors are indifferent between limit and market orders. This depends,* *among other things, on the state of the order book and particularly on* α^0

- α^0 *depends on* l^0 *because* l^0 *determines the flow of executed limit orders and consequently the depth of the order book.*

The equilibrium values $(\alpha^0_{eq}, l^0_{eq})$ *are defined as the unique fixed point solution to this problem of interdependence.*

Proof. Proposition 1 characterizes the model's equilibrium that is analyzed in the paper. It summarizes the results given in propositions 2 to 9. □

The equilibrium strategies and the limit order book dynamics (cf Figure 1) have the following features. In the first phase, for $0 \leq t < \tau$, when the asset common value is equal to v_0, trading occurs because of differences in private values across investors: *lo*'s and *hn*'s trade with each other via the limit order book, *ho*'s and *ln*'s do not trade. In particular buy limit orders are submitted by types *hn* and sell limit orders are submitted by types *lo*. During this phase the dynamics of the limit order market are as follows:

- the limit order book is in a *steady state phase*. The market depths at A^0 and B^0 are both equal to α^0_{eq} and do not vary over time.

- *lo* type investors submit sell limit or market orders using a mixed strategy. They choose to submit a market order at price B^0 with probability m or a limit order at price A^0 with probability $1 - m$. They do not submit sell limit orders at prices higher than A^0.

- *hn* type investors submit buy limit or market orders using a mixed strategy. They choose to submit a market order at price A^0 with probability m or a limit order at price B^0 with probability $1 - m$. They do not submit buy limit orders at prices lower than B^0.

- A sell or buy limit order submitted at price A^0 or B^0 is executed at an idiosyncratic random time according to a Poisson distribution of intensity l^0_{eq}. It is such that *lo*'s and *hn*'s are indifferent between limit and market orders.

Once the asset common value has changed, a *transition phase* starts for the limit order book. This transition phase lasts for a finite duration T. During this phase, for $\tau < t < \tau + T$, trading takes place because the new value of the asset is not in line with market prices, A^0 and B^0, and thus generates a profit opportunity.

- if the new common value is $v_0 + \omega$, investors of type *lo* cancel their sell limit orders, if they have any, and resubmit them at a higher price A^u; investors of type *ho* stay out of the order book; investors of types *hn* and *ln* send buy market orders to execute against stale limit orders at price A^0 and then behave according to their new type, *ho* or *lo*.

- if the new common value is $v_0 - \omega$, investors of type *hn* cancel their buy limit order, if they have any, and resubmit them at a lower price B^d; investors of type *ln* stay out of the order book; investors of types *ho* and *lo* send sell market orders to execute against stale limit orders at price B^0 and then behave according to their new type, *hn* or *ln*.

When all of the limit orders that could potentially be picked off have been executed or cancelled, the transition phase is over. Then trading occurs again because of differences in *private values* across investors. During this last phase, *ho*'s and *ln*'s do not trade, and *lo*'s and *hn*'s trade with each other via the limit order book. The limit order book *converges to a steady state phase* that has the same features as the first phase except that there is no uncertainty for the future common value, $\mu = 0$. Bid and ask prices are either A^u and B^u or A^d and B^d depending on the previous change in the common value.

4.2 Other equilibria

There are other equilibria than the one described above. First there is the class of equilibria to which the symmetric equilibrium belongs. Each of these equilibria is characterized by the three pairs of prices, (A^0, B^0), (A^u, B^u) and (A^d, B^d). The equilibrium strategy is qualitatively the same and differs from the symmetric equilibrium strategy in the values that the equilibrium parameters (α^0_{eq}, l^0_{eq},...) take.

There are also some other classes of equilibria but they are not easy to find. Indeed in order to solve for the equilibrium of this game one must proceed by guess and check. The first step is to conjecture equilibrium strategies for all agents. The easiest thing is to assume that all agents have the same strategy. Given this strategys it is possible to determine the dynamics of the limit order book. The last step is then to check that it is not profitable to operate a one-shot deviation from the conjectured strategy for any type at any point in the game while other agents are playing the conjectured strategy. Solving the problem in that way is difficult. Defining the set of all equilibria is even harder.

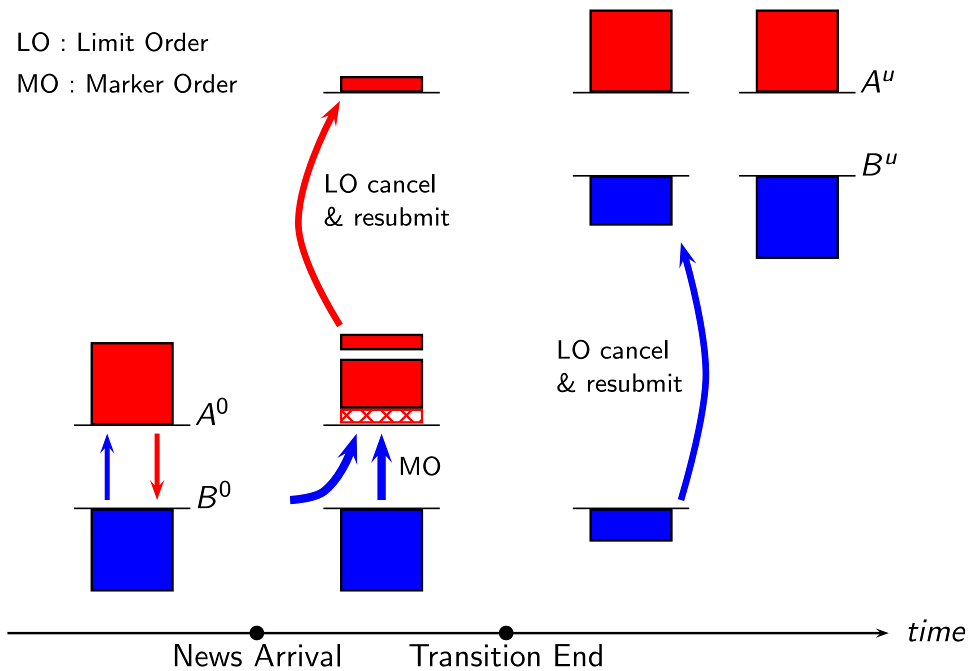

Figure 1: Dynamics of the limit order book following the arrival of "good" news

An example of another equilibrium is the *empty limit order book equilibrium*. It has the following features:

- For $0 \leq t < \tau$, when the asset common value is equal to v_0, trading takes place because of differences in private values across investors: lo's and hn's trade with each other via the limit order book while ho's and ln's do not trade. Investors coordinate on a trading price P at which ho's and ln's send (marketable) limit orders. The buy and sell order flows due to lo's and hn's are exactly equal, which implies that their limit orders are immediately executed and that the limit order book is always empty.

- For $t > \tau$ the same type of trading dynamics take place at price P^u if $v = v_0 + \omega$ and at P^d if $v = v_0 - \omega$

In this type of equilibrium, there is no transition phase. The trading price immediately adjusts when the common value changes. Moreover, this equilibrium reaches the maximum welfare possible in the model. Indeed, it is such that investors of type ho and ln have incentives to trade immediately and thus do not stay with their current position more than an infinitesimal period of time. In the end, all of the asset supply is owned by investors with a high private value, which is socially optimal.

In this paper, I focus on the first class of equilibria, which is more realistic than the second one. Moreover, in the empty limit order book equilibrium, investors coordinate with each other to trade at a price that they do not observe when they observe the limit order book since it is empty. When the limit order book is not empty, investors can figure out what the equilibrium prices are by observing the state of the limit order book.

5 Limit order book in a steady state

In this section, I describe the equilibrium strategy and the state of the limit order book in the first phase of the game. During this phase the limit order book is in a steady state. This phase ends at the time when news arrives. In the description of the steady state phase, I take the bid and ask prices at which trading occurs, (A, B), as first undetermined. At some point, I focus on the pair of prices (A^0, B^0) that defines the symmetric equilibrium. In this way, I separate what is specific to the symmetric equilibrium and what is not.

Definition 1. *A limit order market is in a steady state when the depths displayed in the order book and the different order flows are deterministic and do not change over time.*

This steady state is possible in the model because there is a continuum of investors. Each investor faces idiosyncratic uncertainty about her private value for the asset. She switches from "high" to "low" or "low" to "high" with respect to a Poisson process of intensity ρ. By the law of large numbers applied to the continuum of investors, the share of investors switching from one private value to another is deterministic and equal to $\rho.dt$ at each point in time. For the same reason, the share of investors monitoring the market is deterministic and equal to $\lambda.dt$. Hence the flow of investors who monitor the market is constant over time.

5.1 One-tick market

Proposition 2. *A limit order market in a steady state at equilibrium is necessarily a one-tick market. Its bid-ask spread is equal to the tick of the market,* $A - B = \Delta$*. Moreover liquidity supply is concentrated at best bid and ask prices:*

- *all sell limit orders are sent at price A, generating depth D_A, and there are no sell limit orders at higher prices than A*
- *all buy limit orders are sent at price B, generating depth D_B, and there are no buy limit orders at lower prices than B*

Proof. The proof of proposition 2 relies on a simple argument. In a steady state at equilibrium limit orders and market orders are sent by investors in accordance with an equilibrium strategy. It generates flows of limit and market orders that are constant and deterministic over time so that the steady state holds. Let us consider an investor for whom it is optimal to send a buy market order at A. If there was a reachable price $A < P < B$ it would be profitable to send a limit order at P since it would immediately be hit by the flow of market orders and would get price improvement compared to A. This would contradict the optimality of the strategy. □

This one-tick market result relies on the modelling approach. There is a continuum of investors and a "0 or 1" asset holding constraint. Random idiosyncratic events affect a deterministic share of investors, because of the law of large numbers, which create deterministic flows of orders and cancellations. These flows are finite because of the holding constraint. The key reason for this result is that the flow of market orders is deterministic, continuous and positive. Hence, any limit order that is submitted alone within the bid-spread would be immediately executed. It is also because the instantaneous flow of market orders is infinitesimal and is thus not big enough to move prices. One might think that in a large market where trades take place almost continuously and where market orders are small enough to have no price impact, for instance if robots optimize execution by slicing big orders into small ones, then the occurrence of one-tick bid-ask spreads could be high. Indeed, there would be an incentive to send a limit order within the best quotes rather than a market order because execution would be almost immediate.

5.2 Steady state strategy

When the limit order book is in a steady state trading takes place due to differences in private values across investors. Investors of types *ho* and *ln* do not trade because prices are between the value of owning the asset for an investor with a high private value and that of an investor with a low private value. Given these prices, investors of type *hn* and *lo* are better off if they change their holding status, therefore they trade. If they use market orders, they immediately join the group of "satisfied" investors (*ho* and *ln*). If they use limit orders, they become "satisfied" once their order is executed. The consequence of this strategy is that investors of types *lo* and *hn*, who have monitored the market at least once, have a limit order in the order book. In the steady state, they all have a limit order in the order book since they have all monitored the market at least once.

Proposition 3. *The equilibrium strategy in the steady state phase is defined as follows:*

- *ho: cancel any sell limit order and stay out of the market*
- *hn: send a buy limit or market order as part of a mixed strategy. When she monitors the market she submits a buy market order with probability* $m_A \in [0,1]$*. It is executed at the ask price* A*.*
- *lo: send a sell limit or market order as part of a mixed strategy. When she monitors the market she submits a sell market order with probability* $m_B \in [0,1]$*. It is executed at the bid price* B*.*
- *ln: cancel any buy limit order and stay out of the market.*

In this equilibrium, investors of type *ho* or *ln* do not have a limit order in the order book. When an *ho* type switches to a *lo* type she also monitors the market: either she switches to a *ln* type, if she sends a sell market order, or she remains a *lo* type if she sends a sell limit order. Symmetrically, when an *ln* type switches to a *hn* type, she also monitors the market: either she switches to an *ho* type, if she sends a buy market order, or she remains a *hn* type if she sends a buy limit order. Consequently, at equilibrium, the depths of the order book at prices A and B are equal to the number of investors of types *lo* and *hn* respectively: $D_A = L_{lo}$ and $D_B = L_{hn}$.

5.3 Steady state populations

In a steady state, the aggregate size of each type of population does not change over time. Here, the flow of investors that migrate from the pool of investors with a high private value to the pool of investors with a low private value and the reverse flow must be equal to each other, $\rho(L_{ho} + L_{hn}).dt = \rho(L_{lo} + L_{ln}).dt$. Combined with the constraints that the overall population has a size equal to 1, we obtain that $L_{lo} + L_{ln} = L_{ho} + L_{hn} = \frac{1}{2}$. In addition, the asset supply equal to 1/2 imposes that $L_{lo} + L_{ho} = \frac{1}{2}$. These constraints lead to the following proposition:

Proposition 4. *In a steady state, there is one freedom parameter $\alpha^0 \in \mathbb{R}$ such that the different populations satisfy*

$$L_{ho} = L_{ln} = \frac{1}{2} - \alpha^0, \tag{9}$$

$$L_{hn} = L_{lo} = \alpha^0. \tag{10}$$

It must satisfy the constraints of non-negativity, $\frac{1}{2} - \alpha^0 \geq 0$ and $\alpha^0 \geq 0$.

This freedom parameter α^0 is determined at equilibrium. It is equal to the liquidity supply in the limit order book since the depths are equal to $D_A = L_{lo} = \alpha^0$ and $D_B = L_{hn} = \alpha^0$. Let us emphasize here that proposition 4 is true in any steady state equilibrium and does not depend on the structure of the market. This would remain true in an over-the-counter market framework *à la* Duffie, Gârleanu and Pedersen (2005).

5.4 Micro-level dynamics of the limit order book

At equilibrium, investors of types hn and lo are indifferent between limit and market orders and play a mixed strategy. This implies that both types of orders are submitted in the market. The flows of limit and market orders must be such that the limit order book is in a sustainable steady state. Moreover these flows must be in a steady state as well, that is, constant over time.

The flows of buy market orders and buy limit orders depend on the share, m_A, of investors of type hn who send buy market orders when they monitor the market and on the complementary share, $1 - m_A$, who send buy limit orders. On the sell side, a share, m_B, of investors of type lo send sell market orders when they monitor the market and the remaining share, $1 - m_B$, send sell limit orders.

Ask Side. At each time t, on the ask side of the market, depth is constantly equal to $D_A = L_{lo}$. The flows of order entering or leaving the ask side of the order book are as follows:

- *Outflow due to limit order executions:* execution of buy market orders sent by hn's monitoring the market, $m_A(\lambda L_{hn} + \rho L_{ln}).dt$.

- *Outflow due to limit order cancellations:* investors switching from lo to ho, $\rho L_{lo}.dt$, lo's cancelling their sell limit order to send a sell market order, $m_B \lambda L_{lo}.dt$

- *Inflow due to limit order submissions:* investors switching from ho to lo submitting a sell limit order, $(1 - m_B)\rho L_{ho}.dt$

The steady state condition for the ask side of the book is :

$$\rho L_{lo} + m_A(\lambda L_{hn} + \rho L_{ln}) + m_B(\lambda L_{lo} + \rho L_{ho}) = \rho L_{ho}. \tag{11}$$

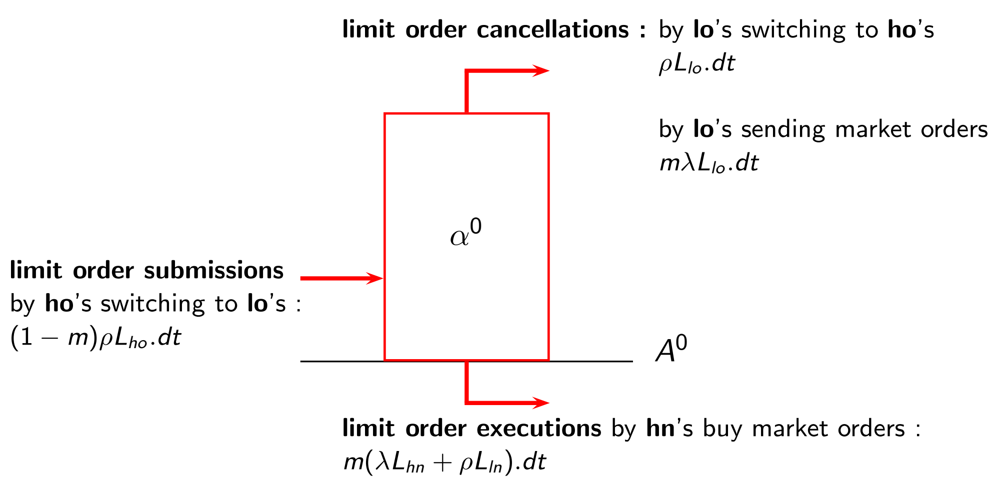

Figure 2: Steady state dynamics of market depth

Bid Side. At each time t, on the ask side of the market the depth is constantly equal to $D_B = L_{hn}$. The flows of orders entering or leaving the bid side of the order book are as follows:

- *Outflow due to limit order executions:* executions of sell market orders sent by lo's monitoring the market, $m_B(\lambda L_{lo} + \rho L_{ho}).dt$.
- *Outflow due to limit order cancellations:* investors switching from hn to ln, $\rho L_{hn}.dt$, hn's cancelling their sell limit order to send a sell market order, $m_A \lambda L_{hn}.dt$
- *Inflow due to limit order submissions:* investors switching from ln to hn submitting a sell limit order, $(1 - m_A)\rho L_{ln}.dt$

The steady state condition is :

$$\rho L_{hn} + m_B(\lambda L_{lo} + \rho L_{ho}) + m_A(\lambda L_{hn} + \rho L_{ln}) = \rho L_{ln}. \tag{12}$$

5.5 Execution rate and liquidity provision

At any time t in the steady state phase, the flow of market orders hits a share of limit orders in the order book. Because of the Pro-Rata execution rule, all limit orders on the same side of the book are equally likely to be executed. Between t and $t + dt$ the probability of execution is equal to the ratio of the flow of market orders over the market depth.

For instance, on the ask side, the flow of market orders is equal to $m_A(\lambda L_{hn} + \rho L_{ln}).dt$ and market depth is equal to $D_A = L_{lo}$. Hence the instantaneous probability of execution is equal to

$$l_A.dt = \frac{m_A(\lambda L_{hn} + \rho L_{ln})}{L_{lo}}.dt. \tag{13}$$

l_A is the execution rate for sell limit orders. In the same way, we can define the execution rate for buy limit orders as $l_B = \frac{m_B(\lambda L_{lo} + \rho L_{ho})}{L_{hn}}$.

Compensation for providing liquidity. When an investor submits a limit order instead of a market order, she chooses her execution price but forgoes execution immediacy and eventually takes the risk of being picked off when the asset common value changes. Providing liquidity therefore requires some compensation for risk-taking. This compensation is obtained through an appropriate execution delay for limit order execution. More precisely, the execution rates l_A and l_B must incentivize liquidity provision via limit orders. At equilibrium these execution rates are such that market and limit orders are equally profitable for types hn and lo. This mechanism appears clearly in investors' value functions (see next sub-section).

Once execution rates and the state of the limit order book are defined at equilibrium, the mixed strategies are perfectly defined as well. For instance, the probability of sending a sell market order for an investor of type lo is equal to $m_B = \frac{L_{hn}}{\lambda L_{lo} + \rho L_{ho}} l_B$.

Steady state liquidity provision. By incorporating the execution rates, the two steady state equations 11 and 12 can be rewritten as

$$\rho L_{hn} + l_B L_{hn} + l_A L_{lo} = \rho L_{ln} \tag{14}$$

$$\rho L_{lo} + l_A L_{lo} + l_B L_{hn} = \rho L_{ho} \tag{15}$$

These equations are in fact the same and define the value of the steady state populations. Indeed, it pins down the value of the depth parameter α^0,

$$\alpha^0 = \frac{1}{2} \frac{\rho}{2\rho + l_A + l_B} \tag{16}$$

The aggregate properties of the limit order book in the steady state phase are completely characterized by α^0 and the execution rates l_A and l_B. Indeed, they define the steady state populations, market depth and aggregate order flows in the limit order book.

5.6 Value functions

The equilibrium strategy generates a system of equations that defines the different value functions for each type of investor. Here I only provide the value functions for investors of types ho and hn. The cases of investors of types ln and lo are very similar.

Type ***ho*****.** At time t, an ho type investor is out of the market and receives a utility flow $v_0.dt$. In the next instant, she may switch to the lo type with probability $\rho.dt$ and obtain the value function V_{lo}. She may also be affected by a change in the asset common value which happens with probability $\mu.dt$. Her value function V_{ho} is defined as follows:

$$V_{ho} = v_0.dt + (1 - r.dt) \left[\rho.dt V_{lo} + \mu.dt \frac{1}{2} (V^u_{ho}(0) + V^d_{ho}(0)) + (1 - (\rho + \mu)dt) V_{ho} \right]. \tag{17}$$

Equation 17 can be simplified and rewritten as:

$$V_{ho} = \frac{v_0 + \rho V_{lo} + \frac{\mu}{2}[V^u_{ho}(0) + V^d_{ho}(0)]}{r + \rho + \mu}. \tag{18}$$

The term $\frac{\mu}{2}[V^u_{ho}(0) + V^d_{ho}(0)]$ corresponds to the average value function of an *ho* type investor after the asset common value has changed, whether upwards or downwards. These are the values of being an *ho* type at the beginning, "time 0", of the transition phase.

Type hn. At time t, an hn type investor monitors the market with probability $\lambda.dt$. She sends a buy market order with probability m_A or a limit order with probability $1 - m_A$. Sending a buy market order at price A provides her with the value function $V_{ho} - A$. Indeed, she gets execution immediacy by trading at the ask price A and instantaneously switches to type *ho*. Sending a buy limit order at price B provides her with the value function V_{hn-B} defined as follows:

$$V_{hn-B} = \frac{\rho V_{ln-out} + m_A\lambda(V_{ho} - A) + l_B(V_{ho} - B) + \frac{\mu}{2}[V^u_{hn-B}(0) + V^d_{hn-B}(0)]}{r + \rho + l_B + m_A\lambda + \mu}. \tag{19}$$

Once the limit order has been submitted, several events can occur: either the investor's type changes with intensity ρ and becomes *ln*, or the investor monitors the market again with intensity λ and cancels her limit order to send a market order with probability m_A, or the limit order is executed with intensity l_B. Each of these events corresponds to a change in the utility function and defines the value function of submitting a limit order. Type *hn* becomes indifferent between limit and market orders if and only if $V_{hn-B} = V_{ho} - A$, that is, if sending a market order or a limit order yields the same expected profit. When this condition is fulfilled, the value function of an *hn* type is defined as $V_{hn} = V_{hn-B} = V_{ho} - A$ and can be rewritten as:

$$V_{hn} = \frac{\rho V_{ln} + l_B(V_{ho} - B) + \frac{\mu}{2}[V^u_{hn-B}(0) + V^d_{hn-B}(0)]}{r + \rho + l_B + \mu}. \tag{20}$$

The indifference conditions for an *hn* type investor define the required values for l_B (similarly the indifference conditions of a *lo* type investor define l_A). It is easy to check that this value does not depend on the mixed strategy m_B. Actually, as mentioned above, these parameters adjust to make the equilibrium possible. Typically l_B depends on value functions in the transition phase. These value functions depend on α^0 since the level of liquidity provision affects the duration of the transition phase, among other things. Let us recall that α^0 corresponds to the depth of the limit order book and that the transition phase lasts until this depth has been completely executed or removed. Overall, l_B depends on α^0 and vice versa. Solving for the game's equilibrium is equivalent to solving this fixed point problem in the first steady state phase. It also requires solving the sub-game starting with the transition phase, after the asset common value has changed, to obtain the corresponding value functions.

5.7 Steady state in the symmetric equilibrium

In the initial steady state of the symmetric equilibrium, investors trade at bid and ask prices A^0 and B^0, defined by equation 8 of proposition 1. The symmetry of this equilibrium is such that the trade-off between limit orders and market orders is quantitatively the same on both sides of the market. The execution rates that make investors indifferent between limit and market orders are the same for sell and buy orders:

$$l_{A^0} = l_{B^0} = l^0. \tag{21}$$

In the symmetric equilibrium, investors of type *lo* or *hn* use the same mixed strategies to choose between limit or market orders. The probabilities m_A and m_B are both equal to m which is equal to

$$m = \frac{\alpha^0 l^0}{\lambda\alpha^0 + \rho\left(\frac{1}{2} - \alpha^0\right)} \tag{22}$$

The steady state condition for the limit order book (equation 16), in the specific case of the symmetric equilibrium, implies that the equilibrium execution rate l^0 depends on the depths of the order book, measured by α^0 in the following way:

$$l^0 = \rho\left(\frac{1}{4\alpha^0} - 1\right) = g(\alpha^0). \tag{23}$$

The execution rate implied by this formula is infinite for $\alpha^0 = 0$ ($g(0) = \infty$) and nil for $\alpha^0 = \frac{1}{4}$ ($g(1/4) = 0$).

Proposition 5. *The execution rate, l^0, that makes investors of types lo and hn indifferent*

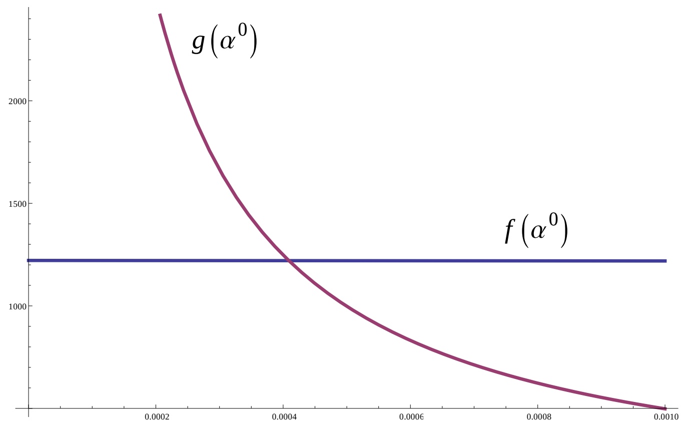

Figure 3: Plot of f and g in function of α^0 ($\lambda = 100,\ \mu = 50,\ r = 1,\ \rho = 2,\ \Delta = 1,\ \delta = 10,\ \omega = 50$).

between limit and market orders is a function of α^0, $l^0 = f(\alpha^0)$. *For* $\alpha^0 = 0$ *and* $\alpha^0 = \frac{1}{4}$, f *is finite and positive. Moreover* f *is decreasing with respect to* α^0,

$$\frac{\partial f}{\partial \alpha^0} < 0. \tag{24}$$

There is a unique $\alpha^0_{eq} \in [0, 1/4]$ *such that* $f(\alpha^0_{eq}) = g(\alpha^0_{eq})$. *The intersection point defines the equilibrium values,* α^0_{eq} *and* l^0_{eq} *(cf Figure 3).*

Proposition 5 ensures that once the trading prices $(A^0, B^0, A^u, B^u, A^d, B^d)$ have been set then the equilibrium aggregate state of the limit order market is unique. Only one level of market depth can be an equilibrium one.

5.8 Steady state equilibria without uncertainty

When there is no uncertainty, $\mu = 0$, the equilibrium dynamic of the limit order market is a steady state dynamic that lasts for the whole duration of the game. In this specific case, the indifference conditions can be solved in closed form. And we can generalize the analysis in principle to any pair of bid and ask prices (A, B), not only the symmetric one. Let us first reconsider the indifference conditions required for an equilibrium to hold.

An hn type is indifferent between limit and market orders when equation 20 holds. In the specific case where $\mu = 0$ and by replacing V_{hn} with $V_{ho} - A$, since they are equal, we obtain

$$(r + \rho + l_B)(V_{ho} - A) = \rho V_{ln} + l_B(V_{ho} - B). \tag{25}$$

Similarly a *lo* type is indifferent between limit and market orders at the following condition,

$$(r + \rho + l_A)(V_{ln} + B) = v_0 - \delta + \rho V_{ho} + l_A(V_{ln} + A). \tag{26}$$

Proposition 6. *When there is no uncertainty, the steady state equilibrium is defined by the following values:*

$$l_B = \frac{v_0 - rA - \rho\Delta}{\Delta}, \tag{27}$$

$$l_A = \frac{rB - \rho\Delta - (v_0 - \delta)}{\Delta}, \tag{28}$$

$$V_{ho} = \frac{1}{r}\frac{1}{2}(v_0 - \rho\Delta) + \frac{1}{r + 2\rho}\frac{1}{2}(v_0 + \rho(A + B)), \tag{29}$$

$$V_{ln} = \frac{1}{r}\frac{1}{2}(v_0 - \rho\Delta) - \frac{1}{r + 2\rho}\frac{1}{2}(v_0 + \rho(A + B)), \tag{30}$$

$$V_{hn} = V_{ho} - A, \tag{31}$$

$$V_{lo} = V_{ln} + B. \tag{32}$$

The equilibrium bid and ask prices, B and A, verify the inequalities

$$\frac{v_0}{r} - \frac{\delta}{r} + \frac{\rho}{r}\Delta \leq B < A \leq \frac{v_0}{r} - \frac{\rho}{r}\Delta, \tag{33}$$

and the one-tick market property, $A - B = \Delta$. And the equilibrium populations are characterized by the value

$$\alpha^0_{eq} = \frac{1}{2}\frac{\rho\Delta}{\delta - r\Delta}. \tag{34}$$

Symmetric equilibrium. The particular pair of prices on which I focus in this paper defines the symmetric case. This is the case where the term of limit order vs. market order trade-off is the same on both sides of the market. The equilibrium prices (A^0, B^0) make the execution rates equal, $l_{A^0} = l_{B^0}$. In the particular setting, in which $\mu = 0$, the value of the execution rate is $l_{A^0} = l_{B^0} = l^0_{eq} = \frac{\delta - (r+2\rho)\Delta}{2\Delta}$. The execution rate depends on δ, the difference between the low and high private value, which measures gains from trade. The bigger it is the less investors are willing to wait and thus require a higher execution rate. This execution rate is also negatively impacted by the tick size since a high tick size corresponds to a high trading cost for market orders. It makes investors more willing to wait and trade with a limit order to save the bid-ask spread.

Monitoring intensity irrelevance. An interesting feature of this equilibrium is that aggregate outcomes, such as α^0_{eq}, do not depend on λ, the monitoring intensity. This is an expected outcome of the model since trades occur because of differences in private values and because these private values are monitored continuously. This suggests that market monitoring has a limited role in a stable market. More specifically, market monitoring plays a role when liquidity supply is, for instance, cyclical, such as in Foucault et al. [2009]. In my model, there is no cycle since order flows are such that the order book is steady.

Remark 1. *The assumption 5, $\delta - (r + 2\rho)\Delta > 0$, is necessary to ensure that the interval $\left[\frac{v_0}{r} - \frac{\delta}{r} + \frac{\rho}{r}\Delta, \frac{v_0}{r} - \frac{\rho}{r}\Delta\right]$ is non-empty and larger than Δ.*

To understand inequality 33, we can look at the interval prices $\left[\frac{v_0}{r} - \frac{\delta}{r}\frac{r+\rho}{r+2\rho}, \frac{v_0}{r} - \frac{\delta}{r}\frac{\rho}{r+2\rho}\right]$. First, let us emphasize that this interval is a subset of all possible equilibrium prices - it is included in $\left[\frac{v_0}{r} - \frac{\delta}{r} + \frac{\rho}{r}\Delta, \frac{v_0}{r} - \frac{\rho}{r}\Delta\right]$. This inclusion is a consequence of $\delta - (r + 2\rho)\Delta > 0$. $\frac{v_0}{r} - \frac{\delta}{r}\frac{r+\rho}{r+2\rho}$ is the value, for an investor with a low private value, to hold the asset forever and $\frac{v_0}{r} - \frac{\delta}{r}\frac{\rho}{r+2\rho}$ is the value, for an investor with a high private value, to hold the asset forever. These are the reserve values of these investors when they hold the asset. In the case where an owner with a low private value and a non-owner with a high private value meet only once, and leave the market afterwards, then the trading price has to be in this interval for the two investors to trade.

In the steady state equilibrium of the limit order market, trading also takes place between owners with a low private value and non-owners with a high private value. However, the range of trading prices is wider than the difference between the two reserve values because investors can trade more than once.

6 Limit order book in the transition phase

When there is uncertainty, $\mu > 0$, the first steady state phase does not last forever and is followed by a transition phase. Preceding the beginning of the transition phase, the world is in the state $\zeta = 0$. The transition phase starts when the asset common value changes. It corresponds to the public news arrival event. It occurs at some point in time τ that follows a Poisson distribution, $\mathcal{P}(\mu)$. At times $t > \tau$ the state of the world is either $\zeta = u$ (up) with $v_u = v_0 + \omega$ or $\zeta = d$ (down) with $v_d = v_0 - \omega$ with equal probability. T is the the duration of the transition phase and it is the same in both states of the world.

6.1 Transition phase strategy

Once the asset common value has changed, to a high level v_u for instance, investors of types hn and ln who do not own the asset turn into arbitrageurs. It is profitable for them to buy the asset while it is tradable at a low price, A^0, and to resell it at a higher price, A^u, later. The price A^u will also be the ask price in the phase that will follow the transition phase.

Proposition 7. *After the asset common value has changed, during the transition phase, the equilibrium strategy is as follows:*

- *In the case $\zeta = u$, for $\tau < t < \tau + T$: investors coordinate on a pair of future ask and bid prices (A^u, B^u); investors of type lo cancel any sell limit order that is not at price A^u and submit a limit order at price A^u; investors of type ho cancel any sell limit order and stay out of the market; investors of type ln send a buy market order and immediately follow the strategy of investors of type lo; investors of type hn send a buy market order and immediately follow the strategy of investors of type ho.*

- *In the case $\zeta = d$, for $\tau < t < \tau + T$: investors coordinate on a pair of future ask and bid prices (A^d, B^d); investors of type hn cancel any buy limit order that is not at price B^d and submit a limit order at price B^d; investors of type ln cancel any buy limit order and stay out of the market; investors of type ho send a sell market order and immediately follow the strategy of investors of type hn; investors of type lo send a sell market order and immediately follow the strategy of investors of type ln.*

6.2 Limit order book dynamics in the transition phase

Before the transition phase begins the limit order book is filled with limit orders submitted during the previous steady state phase. The initial market depths at best ask and bid prices A^0 and B^0 are noted D_{A^0} and D_{B^0}. Both are equal to α^0. During the transition phase, trading occurs only on one side of the order book. It is the side of the market where posted limit orders offer a profit opportunity. On the other side, limit orders do not offer such an opportunity. Investors with these orders will cancel them and send market orders to hit limit orders on the other side.

For instance, when the asset common value makes a positive jump, $\zeta = u$, sell limit orders submitted at price A^0 offer a profit opportunity to buyers. Indeed A^0 was an equilibrium price when the asset common value was equal to v_0 but is no longer once this value has moved up to $v_0 + \omega$. At time $t > \tau$, market depth on the ask side $D^u_{A^0}(t)$ is meant to disappear. It decreases by virtue of two mechanisms. At each time t, a mass $(\lambda + \rho) D^u_{A^0}(t).dt$ of investors who have a limit order posted at A^0 monitor the market and cancel their limit orders at price A^0. At the same time, a mass $(\lambda + \rho) \times (L_{hn}(t) + L_{ln}(t)).dt = \frac{\lambda+\rho}{2}.dt$ of investors, who do not own the asset, monitor the market and send buy market orders that execute at price A^0. The dynamics of the depth $D^u_{A^0}(t)$ are given by the following proposition:

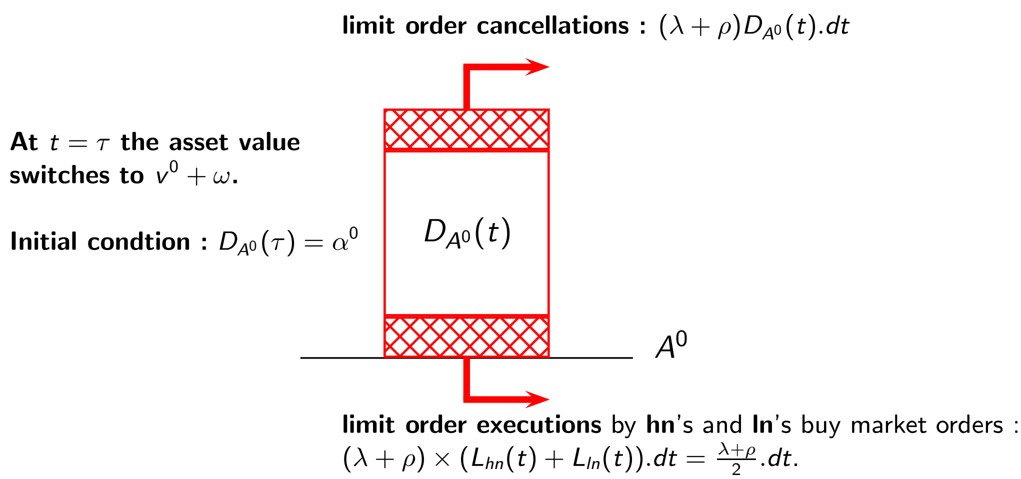

Figure 4: Dynamics of market depth at A^0 following the arrival of "good" news

Proposition 8. *After the common value has changed, during the transition phase,*

- *when* $\zeta = u$*, market depth at price* A^0 *varies as*

$$D^u_{A^0}(t) = D(t) = -\frac{1}{2} + \left(\alpha^0 + \frac{1}{2}\right) e^{-(\rho+\lambda)(t-\tau)} \tag{35}$$

- *when* $\zeta = d$*, market depth at price* B^0 *varies as* $D^d_{B^0}(t) = D(t)$ *as well.*

- *In both cases, the transition phase has the same duration* T *with*

$$T = \frac{1}{\rho+\lambda} \ln(1 + 2\alpha^0). \tag{36}$$

Transition phase in the symmetric equilibrium. The symmetric equilibrium is the equilibrium at which investors will trade on future bid and ask prices (A^u, B^u) in state $\zeta = u$, or (A^d, B^d) in state $\zeta = d$ (defined by equation 8 of proposition 1), in the phase that follows the transition phase. In section 8, I describe in more detail the underlying trading mechanism in the transition phase. It has interesting empirical implication regarding the impact of news arrival frequency or market monitoring on the speed of price adjustment following news arrival. It also makes it possible to disentangle the different roles that limit and market orders play in this price formation process. In particular, it is possible to quantify the effect of news arrival frequency on the share of limit order cancellations and market order executions in the erosion process of initial market depth.

7 After the transition phase : convergence to a steady state without uncertainty

In this section, I describe the strategy and the dynamics of the limit order book that correspond to the last phase of the game, after the transition phase is over. In the last phase of the game, the limit order book converges to a steady state. The asymptotic steady state of the last phase has the same type of strategy as in the steady state of the first phase. Trading takes place at prices A^u and B^u if $\zeta = u$ or at A^d and B^d if $\zeta = d$. The main difference is that there is no aggregate uncertainty in the last phase, $\mu = 0$.

Here I present the general case of this dynamic equilibrium that converges to a steady state without uncertainty. In this equilibrium, the trade-off between limit and market orders does not change over time. The different value functions are constant over time and are the same as those in the equilibrium steady state without uncertainty. I use the same notations I used for the limit order book in a steady state without uncertainty ($\alpha^0_{eq},\ A,\ B$...etc).

The game starts at $t = 0$ in a one-tick market where the market depths at prices A and B are $D_A(0)$ and $D_B(0)$. They are constituted by a share of the population $L_{lo}(0)$ and a share of $L_{hn}(0)$ respectively. Investors use their steady state equilibrium strategy described in section 4. The rates at which hn and lo types send market orders, $m_A(t)$ and $m_B(t)$, vary so that the limit order execution rates, l_A and l_B, are constant over time and equal to their value in the steady state equilibrium. In this setting, the dynamics of the different populations are given by the dynamics of the parameter $\alpha(t)$:

$$L_{ho}(t) = L_{ln}(t) = \frac{1}{2} - \alpha(t) \tag{37}$$

$$L_{hn}(t) = L_{lo}(t) = \alpha(t). \tag{38}$$

To fully characterize the level of convergence of the limit order book, we look at how its state is different from the limit steady state. First, in the steady state all investors have positions in line with their optimal strategy. For instance, at the limit $t = \infty$ all lo types have a limit order in the book at price A. In this game, an lo type investor may be out of the market at $t = 0$ and then has to wait for her first market monitoring time to submit a limit order. Hence, $L_{lo}(t) - D_A(t)$ measures the mass of lo types out of the market. At time t, the mass of investors of type lo or hn that are still out of the market are given by the following equations:

$$L_{lo}(t) - D_A(t) = (L_{lo}(0) - D_A(0))e^{-(\lambda+\rho)t}, \tag{39}$$

$$L_{hn}(t) - D_B(t) = (L_{hn}(0) - D_B(0))e^{-(\lambda+\rho)t}. \tag{40}$$

Second, we must measure the difference between the population sizes at time t and their level in the steady state equilibrium. This difference is given by the value $\alpha(t) - \alpha^0_{eq}$. Overall, describing the changes in $\alpha(t)$ is enough to describe the dynamics of the order book since $L_{ho}(t)$, $L_{ln}(t)$, $L_{hn}(t)$, $L_{lo}(t)$, $D_A(t)$ and $D_B(t)$ are fully defined when $\alpha(t)$ is known.

Proposition 9. *The dynamics of the parameter $\alpha(t)$ are given by the following equation:*

$$\alpha(t) = \alpha_{eq} + \left[\alpha(0) - \alpha_{eq}^0 + (l_A\kappa_A + l_B\kappa_B)\frac{1 - e^{-[\lambda-(\rho+l_A+l_B)]t}}{\lambda - (\rho + l_A + l_B)}\right] e^{-(2\rho+l_A+l_B)t} \tag{41}$$

with $\kappa_A = L_{lo}(0) - D_A(0),\ \kappa_B = L_{hn}(0) - D_B(0)$

8 Empirical implications

8.1 Determinants of the liquidity supply before news arrival

In the initial steady state phase when investors decide to supply liquidity or not, they anticipate that news arrival will trigger a transition phase during which their limit order may bear adverse selection risk. The model's parameters influence this risk of being picked off in different ways and thus have an effect on the size of market depth.

Proposition 10. *An increase in μ or ω has a negative impact on α_{eq}^0 (cf. Figure 5)*

$$\frac{\partial \alpha_{eq}^0}{\partial \mu} < 0, \quad \frac{\partial \alpha_{eq}^0}{\partial \omega} < 0. \tag{42}$$

Moreover $\lim_{\mu\to\infty} \alpha_{eq}^0 = 0$.

For a value of μ not too low, an increase in the monitoring rate λ has a positive impact on α_{eq}^0 (cf. Figure 5).

$$\frac{\partial \alpha_{eq}^0}{\partial \lambda} > 0 \tag{43}$$

Prediction 1. *The liquidity supply before news arrival α_{eq}^0*

- *decreases with the frequency of news arrival, μ, or news surprise, ω.*
- *decreases with the monitoring rate λ when μ is not too low*

The comparative statics (42) can be explained by the effect that μ and ω have on the limit order execution rate l^0. All other things being equal, investors are less willing to use limit orders when the volatility of the asset common value, driven by μ and ω, increases. As a consequence, in order to keep investors indifferent between limit and market orders, the execution rate must increase.

$$\frac{\partial l^0}{\partial \mu} > 0, \quad \frac{\partial l^0}{\partial \omega} > 0. \tag{44}$$

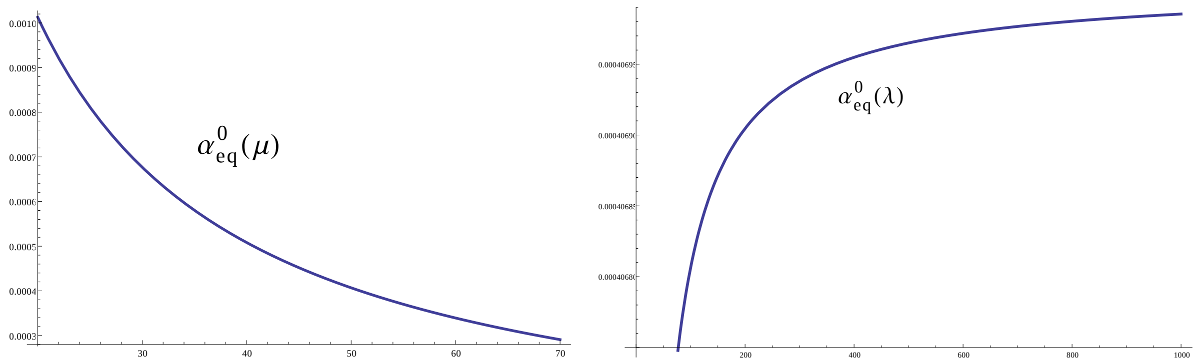

Figure 5: (i) Changes in α^0_{eq} in function $\mu \in [20, 70]$ for $\lambda = 100$ and (ii) changes in α^0_{eq} in function of $\lambda \in [0, 1000]$ for $\mu = 50$ ($r = 1,\ \rho = 2,\ \Delta = 1,\ \delta = 10,\ \omega = 50$).

Then the depth of the order book adjusts downwards because the relation $\alpha^0 = \frac{\rho}{4(\rho+l^0)}$ has to be satisfied.

The effect of λ on α^0_{eq} can also be explained by the fact that l^0 decreases with respect to λ. However this dependence of l^0 on λ arises from two different channels with opposite effects and hence has no intuitive direction. On the one hand, the increase of λ makes investors quicker to cancel limit orders following news arrival. It reduces the picking-off risk and increases the incentive to use limit orders through, then l^0 decreases to maintain the indifference condition. On the other hand, when λ increases, investors can send market orders faster following news arrival, which increases the picking-off risk for limit orders, which ultimately increases l^0. Overall, the effect is ambiguous. Numerical investigations confirm this ambiguity. We can see that for different values of μ, with the same parametrization as in Figure 5, the effect is different. Figure 6 shows that market depth is non-monotonic with respect to λ for $\mu = 0.1$ and decreasing for $\mu = 0.01$. The effect is changed when μ is fairly small. For instance, for $\mu = 1$, α^0_{eq} increases with λ. But overall, if we look at the magnitude of the variation of α^0_{eq} with λ, it is small and even negligible compared to the effect of μ.

The negligible effect of λ on market depth α^0_{eq}, and implicitly on investors' trading strategies, is in line with the intuition that only relative reaction speed should matter in a game where faster traders will earn higher profits following the arrival of new information. In our set-up, an increase in λ corresponds to an increase in the reaction speed of all investors, but their speed remains the same relative to each other. Put differently, monitoring abilities are homogenous across the population.

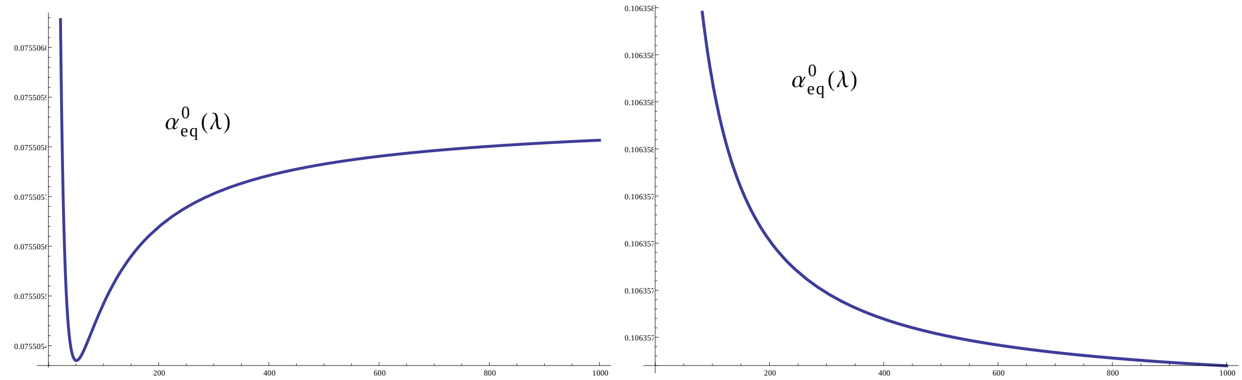

Figure 6: Changes of α^0_{eq} in function of $\lambda \in [0, 1000]$ (i) for $\mu = 0.1$ and (ii) for $\mu = 0.01$ ($r = 1,\ \rho = 2,\ \Delta = 1,\ \delta = 10,\ \omega = 50$).

8.2 Duration between news arrival and price changes

The duration between news arrival and the change in trading prices in the limit order book is the duration of the transition phase.

$$T = \frac{1}{\rho + \lambda} \ln(1 + 2\alpha^0_{eq}) \tag{45}$$

Prediction 2. *For values of λ and μ not too low, the duration of the transition phase T*

- *decreases with the frequency of news arrival, μ, or news surprise, ω.*
- *decreases with the monitoring rate λ.*

Prices in the limit order book reflect the new common value of the asset once there is no arbitrage opportunity left, that is, once limit orders submitted at the initial prices that offered the arbitrage opportunity have disappeared. The population of potential arbitrageurs is fixed. It corresponds to the group of non-owners if the common value goes up and to the group of owners if the common value goes down. Both groups have the same size, $\frac{1}{2}$. Then, in the transition phase, the flow of directional market orders used to take advantage of the arbitrage opportunity is proportional to the rate at which this population monitors the market, $\lambda + \rho$. This flow is equal to $[(\lambda + \rho)/2]dt$ and does not depend on μ or ω. μ and ω only affect the initial market depth α^0_{eq}. The effect of an increase of μ or ω is mechanical: it reduces the initial market depth. Thus, it is consumed and removed faster in the transition phase. The monitoring intensity λ affects both market depth and the flow of market orders.

However, as market depth is a bounded function of λ (and is not very sensitive to λ), the monitoring rate ends up reducing the duration of the transition phase for λ "not too low".

We should note that as soon as the asset holding constraint on investors is independent of μ or ω during the transition phase, the flow of market orders remains independent of these parameters and the result holds. The "0 or 1" asset holding constraint is not key here. However, the holding constraint is necessary, otherwise investors would send infinitely large orders and consume the initial market depth instantaneously .

The fact that λ and μ or ω are independent is less obvious. As in the case of models of limited attention allocation (Peng and Xiong (2006), Van Nieuwerburgh and Veldkamp (2009) Mondria (2010)), investors may choose their λ depending on characteristics of the asset. This calls for further extensions of the model to endogenize the choice of λ.

8.3 Order flow decomposition in the price discovery process

Corollary 1. *In the transition phase, the numbers of limit orders executed and limit orders cancelled are respectively:*

$$LOE = \frac{\ln(1 + 2\alpha^0_{eq})}{2}, \quad LOC = \alpha^0_{eq} - \frac{\ln(1 + 2\alpha^0_{eq})}{2}. \tag{46}$$

The ratio of limit order cancellations to executed limit orders is increasing with respect to α^0:

$$\frac{\partial}{\partial \alpha^0} \frac{LOC}{LOE} > 0 \tag{47}$$

Prediction 3. *In the transition phase, the ratio of limit order cancellations to limit order executions*

- *decreases with the frequency of news arrival, μ, or news surprise, ω.*
- *increases with the monitoring rate λ when μ is not too low.*

The mechanism behind this result is as follows. As mentioned in the previous sub-section, the flow of directional market orders during the transition phase is proportional to $\lambda + \rho$ and does not depend on α^0_{eq}, μ or ω. On the the liquidity supply side, the instantaneous probability for an investor to cancel her limit order is also $(\lambda + \rho).dt$. The mass of these investors is α^0_{eq} at the beginning of the transition phase and equal to $D(t)$ afterwards. Then the flow of limit order cancellations at t during the transition phase is $(\lambda+\rho)D(t).dt$, which depends positively on α_{eq}^{0}. If α_{eq}^{0} is initially larger, at each date the flow of limit order cancellations is larger as well, whereas the flow of market orders is the same. This explains why the number of limit order cancellations increases relative to the number of executions. When α_{eq}^{0} is larger, the transition phase also lasts longer, which explains why the number of market orders during the transition phase can increase as well.

9 Conclusion

This paper models the effect of limited attention on market reactions to news arrival. The limited attention capacity of investors restricts their ability to monitor the market. This imperfect market monitoring delays price adjustments following news arrival. Because of their imperfect ability to monitor news, investors run the risk of being picked off when they supply liquidity with limit orders. When the frequency of news arrival increases, this picking-off risk is amplified and consequently (i) market depth decreases, (ii) prices adjust faster following news arrival and (iii) the number of limit order cancellations in the price formation process decreases relative to the number of limit order executions.

A Proofs

A comprehensive version of the propositions' proofs can be found in the Online Appendix. The Online Appendix also sets out a fundamental lemma which proves that if a strategy cannot be improved with a one-shot deviation then there is no profitable deviation at all. A one-shot deviation from a strategy is a strategy that deviates at the first (upcoming) monitoring time and then follows the initial strategy in the future.

A.1 Proof of proposition 3

In the state $\zeta = 0$ when the limit order book is in the steady state, the value functions are

- for type ho, $V^0_{ho} = V^0_{ho-out}$ with

$$(r+\rho+\mu)V^0_{ho-out} = v_0 + \rho V^0_{lo} + \frac{\mu}{2}V^u_{ho-out}(0) + \frac{\mu}{2}V^d_{ho-out}(0)$$

- for type lo, the value function of sending limit orders at A^0 is

$$(r+\rho+l^0_{A^0}+m_A\lambda+\mu)V^0_{lo-A^0} = v_0-\delta+\rho V^0_{ho}+l^0_{A^0}(V^0_{ln}+A^0)+m_A\lambda(V^0_{ln}+B^0)+\frac{\mu}{2}V^u_{lo-A^0}(0)+\frac{\mu}{2}V^d_{lo-A^0}(0)$$

  Moreover such an investor is indifference between limit and market orders, hence $V^0_{lo-A^0} = V^0_{ln} + B^0$, which allows to define V^0_{lo} by

$$(r+\rho+l^0_{A^0}+\mu)V^0_{lo} = v_0 - \delta + \rho V^0_{ho} + l^0_{A^0}(V^0_{ln}+A^0) + \frac{\mu}{2}V^u_{lo-A^0}(0) + \frac{\mu}{2}V^d_{lo-A^0}(0)$$

- for type ln, $V^0_{ln} = V^0_{ln-out}$ with

$$(r+\rho+\mu)V^0_{ln-out} = \rho V^0_{hn} + \frac{\mu}{2}V^u_{ln-out}(0) + \frac{\mu}{2}V^d_{ln-out}(0)$$

- for type hn, similarly to type lo

$$(r+\rho+l^0_{B^0}+\mu)V^0_{hn} = \rho V^0_{ln} + l^0_{B^0}(V^0_{ho}-B^0) + \frac{\mu}{2}V^u_{hn-B^0}(0) + \frac{\mu}{2}V^d_{hn-B^0}(0)$$

To have indifference for lo and hn types the execution rates of limit orders must be equal to

$$l^0_{B^0} = \frac{1}{\Delta}[v_0 - rA^0 - \rho\Delta + \frac{\mu}{2}(V^u_{ho-out}(0) - A^0 - V^u_{hn-B^0}(0)) + \frac{\mu}{2}(V^d_{ho-out}(0) - A^0 - V^d_{hn-B^0}(0))]$$
$$l^0_{A^0} = \frac{1}{\Delta}[rB^0 - \rho\Delta - (v_0 - \delta) + \frac{\mu}{2}(V^u_{ln-out}(0) + B^0 - V^u_{lo-A^0}(0)) + \frac{\mu}{2}(V^d_{ln-out}(0) + B^0 - V^d_{lo-A^0}(0))]$$

Proposition 5 ensures that, for $B^0 = \frac{1}{r}\left(v_0 - \frac{\delta}{2}\right) - \frac{\Delta}{2}, \quad A^0 = \frac{1}{r}\left(v_0 - \frac{\delta}{2}\right) + \frac{\Delta}{2}$, it is possible to find a (unique) couple (α^0, l^0) such that $l^0 = l^0_{A^0} = l^0_{B^0}$ which a function of α and the relation $l^0 = \rho\left(\frac{1}{4\alpha^0} - 1\right)$.

In the Online Appendix (section E.) we check and provide the other conditions under which it is not profitable to deviate from the strategy described in proposition 3.

A.2 Proof of proposition 4

The steady state is defined by the system

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_{ho} \\ L_{hn} \\ L_{lo} \\ L_{ln} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

First we can check that $L_{ho} = \frac{1}{2},\ L_{hn} = 0,\ L_{lo} = 0,\ L_{ln} = \frac{1}{2}$ is a particular solution of this system. Hence, the general space of solutions of this system is equal to

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + \ker \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + Vect \left[\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right]$$

It proves that there is an $\alpha \in \mathbb{R}$ such that

$$\begin{pmatrix} L_{ho} \\ L_{hn} \\ L_{lo} \\ L_{ln} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + \alpha \times \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

A.3 Proof of proposition 5

To calculate $f(\alpha)$, which is the valuel^0 that makes investors indifferent between limit and market orders we need to calculate

$$
\begin{aligned}
V^u_{ln-out}(0) + A^0 - V^u_{lo-A^0}(0) &= V^d_{ho-out}(0) - B^0 - V^d_{hn-B^0}(0) \\
&= -[\omega - \frac{\delta}{2} + r\frac{\Delta}{2}] \int_0^T \frac{D(t)e^{-rt}}{\alpha} ds \\
&= -[\omega - \frac{\delta}{2} + r\frac{\Delta}{2}] \int_0^T h(t)e^{-(r+\rho+\lambda)t} dt \\
V^d_{ln-out}(0) + B^0 - V^d_{lo-A^0}(0) &= V^u_{ho-out}(0) - A^0 - V^u_{hn-B^0}(0) \\
&= \frac{\omega + \frac{\delta}{2} - r\frac{\Delta}{2}}{r+\rho+\lambda} + \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda} e^{-(r+\rho+\lambda)T}
\end{aligned}
$$

with $D(t)$ given in proposition 8 and $h(t) = \frac{D(t)}{\alpha e^{-(\lambda+\rho)t}} = 1 - \frac{1}{2}\frac{(1-e^{-(\lambda+\rho)t})}{\alpha e^{-(\lambda+\rho)t}} = 1 + \frac{1}{2\alpha} - \frac{1}{2\alpha}e^{(\lambda+\rho)t}$ and $T = \frac{\ln(1+2\alpha)}{\rho+\lambda}$, noting that $h(T) = 0$.

$$
\begin{aligned}
f(\alpha) = &\frac{\delta - (r+2\rho)\Delta}{2\Delta} \\
&+ \frac{1}{2\Delta}\mu \left[-\Delta - (\omega - \frac{\delta}{2} + r\frac{\Delta}{2}) \int_0^T h(t)e^{-(r+\rho+\lambda)t} dt \right] \\
&+ \frac{1}{2\Delta}\mu \left[\frac{\omega + \frac{\delta}{2} - r\frac{\Delta}{2}}{r+\rho+\lambda} + \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda} e^{-(r+\rho+\lambda)T} \right]
\end{aligned}
$$

The derivative of $\int_0^T h(t)e^{-(r+\rho+\lambda)t}dt$ with respect to α is

$$
\frac{\partial T}{\partial \alpha} h(T) e^{-(r+\rho+\lambda)T} + \int_0^T \frac{\partial h}{\partial \alpha}(t) e^{-(r+\rho+\lambda)t} dt = \int_0^T \frac{\partial h}{\partial \alpha}(t) e^{-(r+\rho+\lambda)t} dt > 0
$$

then

$$
\frac{\partial f}{\partial \alpha} < 0
$$

We now want to show that equation $f(\alpha^0) = g(\alpha^0)$ has a unique solution on $[0, 1/4]$. This is equivalent to solving the equation

$$
G(\alpha^0) = \alpha^0 \times (2f(\alpha^0) + 2\rho) = \frac{\rho}{2}
$$

Analysis of function G (detailed version in the Online Appendix Section E.3). Let us prove that $G(\alpha) - \frac{\rho}{2}$ has a unique zero on $[0, 1/4]$.

$$G(\alpha) = \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} + \alpha \times \left\{ \frac{\delta}{\Delta} - r + \frac{\mu}{\Delta}\left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda} \right] \right\}$$
$$- \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha)^{\frac{r}{\rho+\lambda}}} + \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}\frac{\alpha}{(1+2\alpha)^{\frac{r}{\rho+\lambda}+1}}$$

The second derivative of G is the second derivative of

$$H(\alpha) = -\frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha)^{\frac{r}{\rho+\lambda}}} + \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}\frac{\alpha}{(1+2\alpha)^{\frac{r}{\rho+\lambda}+1}}$$
$$= \frac{\mu}{\Delta}\frac{\frac{\rho+\lambda}{r}\frac{\delta - r\Delta}{2} - \rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha)^{\frac{r}{\rho+\lambda}}} - \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha)^{\frac{r}{\rho+\lambda}+1}}$$

The second derivative of H is

$$\frac{\partial^2 H}{\partial \alpha^2} = \frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\left[4\frac{r}{\rho+\lambda}\left(\frac{r}{\rho+\lambda}+1\right)\left[\frac{\rho+\lambda}{r}\frac{\delta - r\Delta}{2} - \rho\Delta\right]\frac{1}{(1+2\alpha)^{\frac{r}{\rho+\lambda}+2}} \right]$$
$$- \frac{\mu}{\Delta}\frac{1}{2(r+\rho+\lambda)}\left[4\left(\frac{r}{\rho+\lambda}+1\right)\left(\frac{r}{\rho+\lambda}+2\right)\left[\frac{\lambda+\rho}{r}\omega - \rho\Delta\right]\frac{1}{(1+2\alpha)^{\frac{r}{\rho+\lambda}+3}} \right]$$

The sign of $\frac{\partial^2 H}{\partial \alpha^2}$ is the sign of

$$S(\alpha) = \frac{r}{\rho+\lambda}\left[\frac{\rho+\lambda}{r}\frac{\delta - r\Delta}{2} - \rho\Delta\right] \times (1+2\alpha) - \left(\frac{r}{\rho+\lambda}+2\right)\left[\frac{\lambda+\rho}{r}\omega - \rho\Delta\right]$$

Since $\omega > \frac{\delta - r\Delta}{2}$ and $0 \le \alpha \le 1/4 < 1$, we have

$$S(\alpha) < \left[\frac{\lambda+\rho}{r}\omega - \rho\Delta\right] \times \left[\frac{r}{\rho+\lambda} \times (1+2\alpha) - \left(\frac{r}{\rho+\lambda}+2\right)\right] < \left[\frac{\lambda+\rho}{r}\omega - \rho\Delta\right] \times \left(2\frac{r}{\rho+\lambda} - 2\right) < 0$$

On $[0, 1/4]$, $\frac{\partial^2 H}{\partial \alpha^2} < 0$ then on $[0, 1/4]$ $\frac{\partial G}{\partial \alpha}$ is either always positive, always negative, or positive and then negative. G can at most cross the $\rho/2$ horizontal line on $[0, 1/4]$ twice. And if $G(1/4) > \frac{\rho}{2}$ this means that it is crossed only once.

$$G(1/4) - \frac{\rho}{2} = \frac{1}{4\Delta}\left[\delta - (r+2\rho)\Delta\right] + \frac{\mu}{\Delta}\left[\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} - \frac{\Delta}{4} + \frac{\delta - r\Delta}{4(r+\rho+\lambda)} \right]$$
$$+ \frac{\mu}{\Delta}\left[\frac{\frac{\rho+\lambda}{r}\frac{\delta - r\Delta}{2} - \rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(3/2)^{\frac{r}{\rho+\lambda}}} - \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{2(r+\rho+\lambda)}\frac{1}{(3/2)^{\frac{r}{\rho+\lambda}+1}} \right]$$

First $\delta - (r+2\rho)\Delta > 0$. Moreover $1 > \frac{1}{(3/2)^{\frac{r}{\rho+\lambda}}} > \frac{2}{3}$. Then

$$G(1/4) - \frac{\rho}{2} > \frac{\mu}{(r+\rho+\lambda)\Delta}\left[\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2} - \frac{(r+\rho+\lambda)\Delta}{4} + \frac{\delta - r\Delta}{4}\right]$$
$$+ \frac{\mu}{(r+\rho+\lambda)\Delta}\left[\frac{\frac{\rho+\lambda}{r}\frac{\delta - r\Delta}{2} - \rho\Delta}{3} - \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{3}\right]$$

then

$$\frac{(r+\rho+\lambda)\Delta}{\mu}\left(G(1/4) - \frac{\rho}{2}\right) > \omega \times \left(\frac{1}{6}\frac{\rho+\lambda}{r}\right) + \frac{\delta}{2} \times \left(-\frac{1}{6}\frac{\rho+\lambda}{r} + \frac{1}{2}\right) + r\Delta \times \left(-\frac{1}{6}\frac{\rho+\lambda}{r} - \frac{1}{2}\right) > 0$$

Because $G(0) = 0$ there is unique $\alpha^0_{eq} \in [0, 1/4]$ such that $G(\alpha^0_{eq}) = \frac{\rho}{2}$. Moreover $\frac{\partial G}{\partial \alpha}(\alpha^0_{eq}) > 0$.

A.4 Proof of proposition 6

type ho**.** An investor of ho type keeps his/her asset until he/she switches to the lo type.

$$(r+\rho)V_{ho} = \rho V_{lo} + v$$

type ln**.** An investor of ln type does not sen any order until he/she switches to the hn type.

$$(r+\rho)V_{ln} = \rho^+ V_{hn}$$

type hn**.** An investor of hn type sends a buy market order with probability m_A or a buy limit order with probability $1 - m_A$. At time t, the outflow of the bid side due to market orders is $m_B(\lambda L_{lo} + \rho L_{ho}).dt$, then the probability for a limit order to be executed at t is $l_B.dt$ with

$$l_B = \frac{m_B(\lambda L_{lo} + \rho L_{ho})}{L_{hn}}$$

When such an investor sends a market order or limit order, they obtain respectively, in term of value function

$$V_{hn} = V_{ho} - A$$

or

$$(r + \rho + l_B + m_A\lambda)V_{hn} = \rho V_{ln} + m_A\lambda(V_{ho} - A) + l_B(V_{ho} - B)$$

In both case V_{hn} must coincide, hence

$$(r+\rho+l_B)(V_{ho}-A)=\rho V_{ln}+l_B(V_{ho}-B)$$

type lo**.** An investor of hn type sends a sell market order with probability m_B or a sell limit order with probability $1-m_B$. For the same reason as for type hn, at time t, the outflow of the ask side due to market order is $m_A(\lambda L_{hn}+\rho^+L_{ln}).dt$, then the probability for a limit order to be executed at t is $l_A.dt$ with

$$l_A=\frac{m_A(\lambda L_{hn}+\rho^+L_{ln})}{L_{lo}}$$

As in the previous case, the value function verifies

$$V_{lo}=V_{ln}+B$$

and

$$(r+\rho+l_A+m_B\lambda)V_{lo}=v-\delta+\rho V_{ho}+m_B\lambda(V_{ln}+B)+l_A(V_{ln}+A)$$

leading to

$$(r+\rho+l_A)(V_{ln}+B)=v-\delta+\rho V_{ho}+l_A(V_{ln}+A)$$

First, by replacing V_{hn} by $V_{ho}-A$ and V_{lo} by $V_{ln}+B$ this is easy to obtain that

$$(r+\rho)V_{ho}-\rho V_{ln}=v+\rho B$$
$$(r+\rho)V_{ln}-\rho V_{ho}=-\rho A$$

and then to get the expression of V_{ho} and V_{ln}.

Replacing V_{ln} by $V_{lo}-B$ and V_{ho} by $V_{hn}+A$ in the equation of indifference between market and limit orders we obtain

$$(r+\rho+l_B)(V_{ho}-A)=\rho(V_{lo}-B)+l_B(V_{ho}-B)$$
$$(r+\rho+l_A)(V_{ln}+B)=v-\delta+\rho(V_{hn}+A)+l_A(V_{ln}+A)$$

which gives

$$v+\rho B-(r+\rho)A=l_B(A-B)$$
$$-\rho A+(r+\rho)B-(v-\delta)=l_A(A-B)$$

l_A and l_B must be positive numbers. Hence

$$v + \rho B - (r+\rho)A = v - \rho\Delta - rA > 0$$

and

$$-\rho A + (r+\rho)B - (v-\delta) = rB - (v-\delta) - \rho\Delta$$

A.5 Proof of proposition 7

When $\zeta = u$. During the transition phase the observed types under the conjecture strategy are $lo - A^u$ (lo with a limit order at price A^u), $lo - A^0$, $ho - out$, $ln - out$, $hn - B^0$ and $hn - B^d$. But $hn - B^0$ and $hn - B^d$ can be grouped under the label $hn - out$ because their limit orders are not executed under the conjecture strategy so they would get the same outcome if they were out of the order book. In this framework we can define the system of ODE's defining the value function of these types:

- $lo - A^u$ stay in the limit order book until they switch of type

$$(r+\rho)V^u_{lo-A^u}(t) = v^u - \delta + \frac{\partial V^u_{lo-A^u}}{\partial t} + \rho V^u_{ho-out}(t)$$

  with $V^u_{lo-A^u}(T^u) = \bar{V}^u_{lo-A^u}$ which is the value for a lo of having a limit order at price A^u once the last dynamic equilibrium is played (the optimal strategy in the last phase).

- $ho - out$ stay out until they switch of type

$$(r+\rho)V^u_{ho-out}(t) = v^u + \frac{\partial V^u_{ho-out}}{\partial t} + \rho^- V^u_{lo-A^u}(t), \quad V^u_{ho-out}(T^u) = \bar{V}^u_{ho-out}$$

- $ln - out$ send a buy market order and immediately behave as their new type: they send a sell limit order and become $lo - A^u$

$$(r+\rho+\lambda)V^u_{ln-out}(t) = \frac{\partial V^u_{ln-out}}{\partial t} + \rho(V^u_{ho-out}(t) - A^0) + \lambda(V^u_{lo-A^u}(t) - A^0), \quad V^u_{ln-out}(T) = \bar{V}^u_{ln-out}$$

- $hn - out$ send a buy market order and immediately behave as their new type

$$(r+\rho+\lambda)V^u_{ln-out}(t) = \frac{\partial V^u_{hn-out}}{\partial t} + \rho(V^u_{lo-A^u}(t) - A^0) + \lambda(V^u_{ho-out}(t) - A^0)$$

with $(r+\rho+\lambda)V^u_{hn-out}(T) = (r+\rho+\lambda)\bar{V}^u_{hn-out} = \rho\bar{V}^u_{ln-out} + \lambda\bar{V}^u_{ln-B^u}$ because as soon as this type contact the market being $hn - B^u$ is optimal after T.

- $lo - A^0$ cancel their limit order or are executed

$$(r+\rho+\lambda+k_{A^0}(t))V^u_{lo-A^0}(t) = v^u - \delta + \frac{\partial V^u_{lo-A^0}}{\partial t} + \rho V^u_{ho-out}(t) + \lambda V^u_{lo-A^u}(t) + k_{A^0}(t)(V^u_{ln-out}(t) + A^0)$$

with $V^u_{lo-A^u}(T) = \bar{V}^u_{ln-out} + A^0$ and the intensity rate for the execution of the limit order

$$k_{A^0}(t) = \frac{(\lambda+\rho)L^u_{hn}(t) + (\lambda+\rho)L^u_{ln}(t)}{D^u_{A^0}(t)} = -(\rho+\lambda) - \frac{\dot{D}^u_{A^0}}{D^u_{A^0}(t)}$$

Now we must check that there is no profitable one-shot deviation for an investor of type lo.

- Instead of staying at A^u a type lo could send a sell market order at price B^0 and get $V^u_{ln-out}(t) + B^0$. This is clearly not profitable given what has been said for type ln

- Another deviation could be to send a limit order at A^0. The corresponding value function is $V^u_{lo-A^0}(t)$ Calling $X(t) = V^u_{lo-A^u}(t) - V^u_{lo-A^0}(t)$ we obtain the ODE

$$(r+\rho+\lambda+k_{A^0}(t))X(t) = \frac{\partial X}{\partial t} + k_{A^0}(t)(V^u_{lo-A^u}(t) - V^u_{ln-out}(t) - A^0)$$

The solution of this ODE is as before

$$X(t) = e^{\int_0^t (r+\rho+\lambda+k_{A^0}(s))ds}[C - \int_0^t k_{A^0}(s)(V^u_{lo-A^u}(t) - V^u_{ln-out}(t) - A^0)e^{-\int_0^s (r+\rho+\lambda+k_{A^0}(l))dl}ds]$$

Then $X(t) \times e^{\int_0^t (r+\rho+\lambda+k_{A^0}(s))ds}$ is decreasing and $X(T) = \bar{V}^u_{lo-A^u} - A^0 - \bar{V}^u_{ln-out} = B^u - A^0 > 0$. $X(t)$ is positive over $[0,T]$. This deviation is not profitable.

- A lo type could send a limit order at any price $A^0 < A < A^u$ The corresponding value function would be defined by

$$\begin{aligned}(r+\rho+\lambda)V(t) &= v^u - \delta + \frac{\partial V}{\partial t} + \rho V^u_{ho-out}(t) + \lambda V^u_{lo-A^u}(t) \\ &= \frac{\partial V}{\partial t} + (r+\rho+\lambda)V^u_{lo-A^u}(t) - \frac{\partial V^u_{lo-A^u}}{\partial t}\end{aligned}$$

that would give

$$V^u_{lo-A^u}(t) - V(t) = C \times e^{(r+\rho+\lambda)t}$$

and this order would be executed at T, then $V^u_{lo-A^u}(T) - V(T) = \bar{V}^u_{lo-A^u} - A - \bar{V}^u_{ln-out} = B^u - A > 0$. The deviation is not profitable.

- Lastly a lo type could stay out. The corresponding value function would be define by

$$\begin{aligned}(r+\rho+\lambda)V(t) &= v^u - \delta + \frac{\partial V}{\partial t} + \rho V^u_{ho-out}(t) + \lambda V^u_{lo-A^u}(t)\\ &= \frac{\partial V}{\partial t} + (r+\rho+\lambda)V^u_{lo-A^u}(t) - \frac{\partial V^u_{lo-A^u}}{\partial t}\end{aligned}$$

that would again gives

$$V^u_{lo-A^u}(t) - V(t) = C \times e^{(r+\rho+\lambda)t}$$

And at T, $V^u_{lo-A^u}(T) - V(T) = \bar{V}^u_{lo-A^u} - \bar{V}^u_{lo-out} > 0$.

The proofs for other types and for the case $\zeta = d$ are similar and provided in the Online Appendix Section D.

A.6 Proof of proposition 8

At $t = \tau$, we must have,

$$D^0_{A^0} = L^0_{lo} = \alpha^0, \quad D^0_{B^0} = L^0_{hn} = \alpha^0$$

When $\zeta = u$, for $t > \tau$, the dynamics of market depth at price A^0 are driven by the fact that investors of lo type, with limit orders at A^0, cancel their limit orders or are executed by market orders that are sent by hn and ln, which gives

$$\frac{\partial D^u_{A^0}}{\partial t} = -(\lambda+\rho)D^u_{A^0}(t) - (\lambda+\rho)L^u_{hn}(t) - (\lambda+\rho)L^u_{ln}(t)$$

since $L^u_{ln}(t) + L^u_{hn}(t) = \frac{1}{2}$, it is equivalent to

$$\frac{\partial D^u_{A^0}}{\partial t} = -(\lambda+\rho)D^u_{A^0}(t) - \frac{1}{2}(\lambda+\rho)$$

The solution of this ODE is,

$$D^u_{A^0}(t) = D(t) = -\frac{1}{2} + \left(\alpha^0 + \frac{1}{2}\right) e^{-(\rho+\lambda)(t-\tau)}$$

The transition phases ends at $T+\tau$ such that $D(T+\tau) = 0$, that is,

$$T = \frac{1}{\rho+\lambda}\ln(1+2\alpha^0)$$

When $\zeta = d$, the proof is similar.

A.7 Proof of proposition 9

The dynamics of the limit order book depths are given by the first order differential equations,

$$\frac{dD_A}{dt} = \rho L_{ho}(t) - \rho D_A(t) - l_A D_A(t) - l_B D_B(t) + \lambda(L_{lo}(t) - D_A(t))$$

$$\frac{dD_B}{dt} = \rho L_{ln}(t) - \rho D_B(t) - l_B D_B(t) - l_A D_A(t) + \lambda(L_{hn}(t) - D_B(t))$$

To obtain the differential equation that drives the dynamic of α, we use the differential equation for $D_A(t)$ and the equalities $D_A(t) = L_{lo}(t) - (L_{lo}(0) - D_A(0))e^{-(\rho+\lambda)t}$. We obtain

$$\begin{aligned}\frac{dL_{lo}(t)}{dt} + (\rho+\lambda)(L_{lo}(0) - D_A(0))e^{-(\rho+\lambda)t} &= \rho L_{ho}(t) - (\rho + l_A)(L_{lo}(t) - (L_{lo}(0) - D_A(0))e^{-(\rho+\lambda)t}) \\ &- l_B(L_{hn}(t) - (L_{hn}(0) - D_B(0))e^{-(\rho+\lambda)t}) \\ &+ \lambda(L_{lo}(0) - D_A(0))e^{-(\rho+\lambda)t}\end{aligned}$$

which gives

$$\frac{dL_{lo}(t)}{dt} = \rho L_{ho}(t) - (\rho+l_A)L_{lo}(t) - l_B L_{hn}(t) - l_A(L_{lo}(0) - D_A(0))e^{-(\rho+\lambda)t} - l_B(L_{hn}(0) - D_B(0))e^{-(\rho+\lambda)t}$$

and then use the fact that $L_{ho}(t) = \frac{1}{2} - \alpha(t)$, $L_{hn}(t) = \alpha(t)$, $L_{lo}(t) = \alpha(t)$. to get the final differential equation

$$\frac{d\alpha}{dt} = \frac{\rho}{2} - [2\rho + l_A + l_B]\alpha(t) + l_A\kappa_A e^{-(\lambda+\rho)t} + l_B\kappa_B e^{-(\lambda+\rho)t}$$

To obtain the general solution to this ODE, we look for the functional form $\alpha(t) = c(t)e^{-(2\rho+l_A+l_B)t}$. Then

$$e^{-(2\rho+l_A+l_B)t}\frac{dc}{dt} = \frac{\rho}{2} + l_A\kappa_A e^{-(\lambda+\rho)t} + l_B\kappa_B e^{-(\lambda+\rho)t}$$

and then

$$\begin{aligned}c(t) = c(0) &+ \frac{\rho}{4\rho + 2l_A + 2l_B} \times (e^{(2\rho+l_A+l_B)t} - 1) - l_A\kappa_A\frac{e^{-[\lambda-(\rho+l_A+l_B)]t} - 1}{\lambda - (\rho + l_A + l_B)} \\ &- l_B\kappa_B\frac{e^{-[\lambda-(\rho+l_A+l_B)]t} - 1}{\lambda - (\rho + l_A + l_B)}\end{aligned}$$

Hence

$$\alpha(t) = \alpha_{eq} + (\alpha(0) - \alpha_{eq})e^{-(2\rho+l_A+l_B)t}$$
$$+ l_A\kappa_A \frac{1 - e^{-[\lambda-(\rho+l_A+l_B)]t}}{\lambda - (\rho + l_A + l_B)} e^{-(2\rho+l_A+l_B)t} + l_B\kappa_B \frac{1 - e^{-[\lambda-(\rho+l_A+l_B)]t}}{\lambda - (\rho + l_A + l_B)} e^{-(2\rho+l_A+l_B)t}$$

Now we check that $0 < \alpha(t) < 1/2$. First we can see that obviously $\alpha(t) > 0$. Since α is converging it can have extrema. Given the ODE that defines α, these extrema must verify

$$[2\rho + l_A + l_B]\alpha(t) = \frac{\rho}{2} + l_A\kappa_A e^{-(\lambda+\rho)t} + l_B\kappa_B e^{-(\lambda+\rho)t}$$

This gives

$$[2\rho + l_A + l_B]\alpha(t) \leq \frac{\rho}{2} + l_A\alpha(0)e^{-(\lambda+\rho^+)t} + l_B\alpha(0)e^{-(\lambda+\rho)t}$$

we can rewrite it as

$$[2\rho + l_A + l_B]\alpha(t) \leq \frac{\rho}{2} + (l_A + l_B)\alpha(0)$$

Hence if $\alpha(0) < 1/4$ then $\alpha(t) < 1/4$.

A.8 Proof of proposition 10

We know that $\frac{\partial G}{\partial \alpha}(\alpha^0_{eq}) > 0$. Since we know that when μ is big then α^0_{eq} is close to zero, we must show that $\frac{\partial G}{\partial \lambda}$ around $\alpha = 0$.

$$G(\alpha) = \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} + \alpha \times \left\{\frac{\delta}{\Delta} - r + \frac{\mu}{\Delta}\left[-\Delta + \frac{\delta - r\Delta}{r+\rho+\lambda}\right]\right\}$$
$$- \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)}\frac{1}{(1+2\alpha)^{\frac{r}{\rho+\lambda}}} + \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}\frac{\alpha}{(1+2\alpha)^{\frac{r}{\rho+\lambda}+1}}$$

Then for $\alpha \to 0$

$$G(\alpha) \sim \frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)} \times \left[1 - 1 + \frac{r}{\rho+\lambda}2\alpha\right]$$
$$+ \alpha \times \left\{\frac{\delta}{\Delta} - r - \mu + \frac{\mu}{\Delta}\frac{\delta - r\Delta}{r+\rho+\lambda}\right\}$$
$$+ \frac{\mu}{\Delta}\frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}\alpha$$

which gives

$$G(\alpha) \sim \alpha \times \left\{ \frac{\delta}{\Delta} - r - \mu + \frac{\mu}{\Delta}\frac{\omega}{r} + \frac{\mu}{\Delta}\frac{\delta - (r+2\rho)\Delta}{2(r+\rho+\lambda)} \right\}$$

which is decreasing with respect to λ.

For the last two derivatives this is sufficient to show that whatever the factor of μ or ω in the function G, it is positive. For μ:

$$\frac{\mu}{\Delta}\frac{\rho+\lambda}{r}\frac{\omega - \frac{\delta}{2} + r\frac{\Delta}{2}}{2(r+\rho+\lambda)}\left[1 - \frac{1}{(1+2\alpha)^{\frac{r}{\rho+\lambda}}}\right] + \alpha \times \frac{\mu}{\Delta}\left\{ \frac{(\lambda+\rho)\frac{\omega}{r} - \rho\Delta}{r+\rho+\lambda}\frac{1}{(1+2\alpha)^{\frac{r}{\rho+\lambda}+1}} - \Delta + \frac{\delta - r\Delta}{r+\rho+\lambda} \right\}.$$

We can see clearly that this also works for ω.

A.9 Proof of corollary 1

The amount of cancellation is given by

$$\begin{aligned}
\int_0^T (\rho+\lambda)D(t)dt &= \int_0^T (\rho+\lambda)\left[-\frac{1}{2} + \left(\alpha^0 + \frac{1}{2}\right)e^{-(\rho+\lambda)t}\right]dt \\
&= -\frac{1}{2}(\rho+\lambda)T + \left(\alpha^0 + \frac{1}{2}\right)(1 - e^{-(\rho+\lambda)T}) \\
&= -\frac{1}{2}\ln(1+2\alpha^0) + \left(\alpha^0 + \frac{1}{2}\right)\left(1 - \frac{1}{1+2\alpha^0}\right)
\end{aligned}$$

The second part of the corollary comes from the the fact that $\ln(1+x)/x$ is increasing.

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